MAT 394 - PS4: Elements of Probability Theory

1.) Suppose that a fair six-sided die is rolled three times and let X_1, X_2 and X_3 be the three numbers obtained, in order of occurrence.

(a) Calculate the probability that $X_1 < X_2 < X_3$.

(b) Calculate the conditional probability that $X_1 = 1$ given that $X_1 < X_2 < X_3$.

2.) Sums of independent random variables.

(a) Suppose that X and Y are independent, integer-valued random variables and let $Z = X + Y$. Explain why the following identity is true:

$$\mathbb{P}(Z = n) = \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k)\mathbb{P}(Y = n - k).$$

(b) Use the result from part (a) to show that if $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$, then $Z = X + Y \sim \text{Binomial}(n + m, p)$. Explain why this result is *intuitively* true without doing any calculations. *Hint:* Interpret the binomial distribution in terms of independent trials.

3.) Use the definition of the mean and variance to show that if X is Poisson-distributed with parameter λ, then $\mathbb{E}[X] = \text{Var}(X) = \lambda$.

4.) Show that if $X = \sigma Z + \mu$, where Z is a standard normal random variable and μ and σ are constants with $\sigma > 0$, then X is a normal random variable with mean μ and variance σ^2. *Hint:* Use a change-of-variables to show that X has the "correct" cumulative distribution function (which you will need to express as an integral).

5.) Suppose that a couple, Michael and Nancy, have two children, and that both Michael's brother and Nancy's father's brother's child (i.e., her first cousin) have cystic fibrosis. If (i) both of these cases are caused by the CFTRΔ508 mutation; (ii) the frequencies of the wild type (WW), carrier (WD) and disease genotypes (DD) in this population are 0.9025, 0.095 and 0.0025, respectively; (iii) mating is random with respect to this mutation, i.e., the genotypes of any two individuals forming a couple are independent of one another; and (iv) neither Michael nor Nancy nor any of their parents or aunts or uncles have cystic fibrosis,

(a) what is the probability that both of Michael's parents are carriers of the disease allele?

(b) what is the probability that Nancy's mother is a carrier of the disease allele? *Hint:* What is the probability that an individual chosen at random is a carrier given that they do not have cystic fibrosis?

(c) what is the probability that Nancy's father is a carrier of the disease allele? *Hint:* Consider all of the possible genotypes of Nancy's father's parents.

(d) what is the probability that Nancy and Michael's first child will have cystic fibrosis?

(e) what is the probability that their second child will have cystic fibrosis if their first child does not?

(f) what is the probability that their second child will have cystic fibrosis if their first child does have this condition?

(g) If neither child has cystic fibrosis and each child has one child with a spouse who does not have cystic fibrosis, what is the probability that neither of Michael and Nancy's two grandchildren will have cystic fibrosis?

Hint: Start by drawing a pedigree showing all of the individuals that are relevant to the problem. Also, denote the wild type and disease alleles by the letters W and D, respectively.