Lecture 20: Sums of Independent Random Variables

1.) **Convolutions**

Definition: If $f, g : \mathbb{R} \to \mathbb{R}$ are two integrable real-valued functions, then the **convolution** of f and g is the real-valued function $f * g : \mathbb{R} \to \mathbb{R}$ defined as

$$\begin{aligned}(f * g)(z) &= \int_{-\infty}^{\infty} f(x)g(z-x)dx \\ &= \int_{-\infty}^{\infty} f(z-x)g(x)dx \; = \; (g * f)(z).\end{aligned}$$

The identity between the first and second line follows from a simple change of variables and shows that convolution is a **commutative** operation: $f * g = g * f$.

Application: In probability theory, convolutions arise when we consider the distribution of sums of independent random variables. To see this, suppose that X and Y are independent, continuous random variables with densities p_x and p_y. Then $X + Y$ is a continuous random variable with cumulative distribution function

$$\begin{aligned}F_{X+Y}(z) &= \mathbb{P}\{X + Y \leq z\} \\ &= \int_{x+y\leq z} p_X(x)p_Y(y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} p_X(x)p_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} F_X(z-y)p_Y(y)dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} p_Y(y)p_X(x)dydx \\ &= \int_{-\infty}^{\infty} F_Y(z-x)p_X(x)dx,\end{aligned}$$

where the expression in the fourth line is the convolution $F_X * p_Y$ and the expression in the sixth line is the convolution $F_Y * p_X$. (**Note:** On p. 252, Ross refers to this as the convolution of F_X and F_Y. This is wrong!) The density of $X + Y$ can then be found by differentiating the CDF, giving

$$\begin{aligned}p_{X+Y}(z) &= \frac{d}{dz}F_{X+Y}(z) \\ &= \int_{-\infty}^{\infty} p_X(z-y)p_Y(y)dy \\ &= \int_{-\infty}^{\infty} p_Y(z-x)p_X(x)dx,\end{aligned}$$

which is equal to the convolution of the density functions of X and Y:

$$p_{X+Y}(z) = (p_X * p_Y)(z) = (p_Y * p_X)(z).$$

2.) **Sums of uniform RVs**

Example (Ross, 3a) If X, Y are independent $U(0,1)$-distributed random variables, then the density of $X+Y$ is

$$p_{X+Y}(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 < z \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

This is known as the **triangular distribution**.

3.) **Sums of gamma RVs**

Proposition 3.1 If X and Y are independent gamma-distributed RVs with parameters (s, λ) and (t, λ), then $X+Y$ is also a gamma-distributed random variable with parameters $(s+t, \lambda)$.

Proof: The result follows from the calculation

$$\begin{aligned} p_{X+Y}(z) &= p_X * p_Y(z) \\ &= \frac{1}{\Gamma(s)\Gamma(t)} \int_0^z \lambda e^{-\lambda(z-x)} \Big(\lambda(z-x)\Big)^{s-1} \lambda e^{-\lambda x} (\lambda x)^{t-1} dx \\ &= C e^{-\lambda z} z^{s+t-1}, \end{aligned}$$

where C is a constant. However, since p_{X+Y} is a density, we know that it integrates to 1 and so we can calculate

$$\begin{aligned} C &= \left(\int_0^\infty e^{-\lambda z} z^{s+t-1} dz \right)^{-1} \\ &= \frac{\lambda^{s+t}}{\Gamma(s+t)}. \end{aligned}$$

This shows that

$$p_{X+Y}(z) = \frac{\lambda e^{-\lambda z} (\lambda z)^{s+t-1}}{\Gamma(s+t)},$$

as claimed.

□

General result: By induction, it follows that if $X_1, \cdots, X_n$ are independent gamma-distributed random variables with parameters (t_i, λ), then the sum $X = X_1 + \cdots + X_n$ is a gamma-distributed RV with parameters $\left(\sum_{i=1}^n t_i, \lambda\right)$.

Example (Ross, 3b): Because the exponential distribution with parameter λ is the same as the gamma distribution with parameters $(1, \lambda)$, it follows that if $X_1, \cdots, X_n$ are independent exponential RVs all with parameter λ, then the sum $X = X_1 + \cdots + X_n$ is a gamma RV with parameters (n, λ).

4.) **Sums of normal RVs**

Proposition 3.2: If $X_1, \cdots, X_n$ are independent normal RVs with parameters $(\mu_i, \sigma_i^2), i = 1, \cdots, n$, then their sum $X = X_1 + \cdots + X_n$ is a normal RV with parameters $\sum_{i=1}^n \mu_i$ and $\sum_{i=1}^n \sigma_i^2$.

Proof: It suffices to consider the case $n = 2$, since the full result then follows by induction on n. Also, because $X_i - \mu_i$ is normally distributed with parameters $(0, \sigma_i^2)$, we may assume that $\mu_1 = \mu_2 = 0$.

Then, using the convolution formula, we see that the density of $X + Y$ is

$$\begin{aligned}
p_{X+Y}(z) &= \int_{-\infty}^{\infty} p_X(z-x)p_Y(x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(z-x)^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{x^2}{2\sigma_2^2}\right) dx \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{z^2}{2\sigma_1^2}\right) \int_{-\infty}^{\infty} \exp\left(-\left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right)x^2 + \frac{z}{\sigma_1^2}x\right) dx \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{z^2}{2\sigma_1^2}\right) \exp\left(2\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1} \frac{z^2}{4\sigma_1^4}\right) \\
&\qquad \times \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\left(x - \left(\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2}\right)^{-1} \frac{z}{\sigma_1^2}\right)^2\right] dx \\
&= \frac{1}{2\pi\sigma_1\sigma_2}(2\pi)^{1/2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1/2} \exp\left(-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}\right) \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left(-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}\right),
\end{aligned}$$

which shows that $X_1 + X_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.
□

5.) **Discrete Convolutions**

If X and Y are independent integer-valued random variables with probability mass functions p_X

and p_Y, then $X+Y$ is also an integer-valued random variable with probability mass function

$$\begin{aligned} p_{X+Y}(n) &= \mathbb{P}\{X+Y=n\} \\ &= \sum_k \mathbb{P}\{X=k, Y=n-k\} \\ &= \sum_k \mathbb{P}\{X=k\}\mathbb{P}\{Y=n-k\} \\ &= \sum_k p_X(k)p_Y(n-k). \end{aligned}$$

The expression in the last line of this series of equations can be seen as a discrete convolution.

Example (Ross, 3e): If X and Y are independent Poisson RVs with parameters λ_1 and λ_2, then $X+Y$ is a Poisson RV with parameter $\lambda_1+\lambda_2$.

Proof: Using the discrete convolution formula (and noting that X and Y are both non-negative), the probability mass function of $X+Y$ is

$$\begin{aligned} p_{X+Y}(n) &= \sum_{k=0}^{n} p_X(k)p_Y(n-k) \\ &= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} \\ &= e^{-(\lambda_1+\lambda_2)}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k(1-\lambda_2)^{n-k} \\ &= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}, \end{aligned}$$

which is also the probability mass function for the Poisson distribution with parameter $\lambda_1+\lambda_2$.
□

General result: Using induction, we can show that if $X_1, \cdots, X_n$ are independent Poisson RVs with parameters $\lambda_1, \cdots, \lambda_n$, respectively, then the sum $X = X_1 + \cdots + X_n$ is a Poisson RV with parameter $\lambda_1 + \cdots + \lambda_n$.