Lecture 15: Expectation; The Uniform Distribution

1. Expectation

**Definition:** Let \( X \) be a continuous RV with density function \( p(x) \). Then the **expected value** of \( X \) (also called the **mean** or **expectation** of \( X \)) is defined to be

\[
E[X] = \int_{-\infty}^{\infty} xp(x)dx,
\]

provided that the integral on the right-hand side exists. Thus, as with discrete random variables, the expected value of a continuous random variable can be thought of as a weighted average of the values that the random variable can take, where the weights are provided by the distribution of the variable.

**Remark:** The expected value of a random variable \( X \) may be equal to \( \infty \) or \(-\infty\). For example, suppose that \( X \) has the following density

\[
p(x) = \begin{cases} 
\frac{1}{x^2} & \text{if } x > 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Notice that \( \int_{-\infty}^{\infty} p(x)dx = 1 \), so \( p(x) \) is a legitimate probability density. Nonetheless, the expected value of \( X \) is \( \infty \) since

\[
E[X] = \int_{1}^{\infty} xp(x)dx = \int_{1}^{\infty} \frac{dx}{x} = \infty.
\]

**Expectations of functions:** As in the discrete case, the expectation of a real-valued function of a random variable can be evaluated directly in terms of the distribution of the random variable (cf. the Law of the Unconscious Statistician). Although this result is true in general, we will only prove it under the assumption that \( f(X) \) is non-negative. Our proof will make use of the following lemma.

**Lemma (Ross, 2.1):** If \( Y \) is a nonnegative random variable with density \( p(y) \), then

\[
E[Y] = \int_{0}^{\infty} P(Y > y)dy.
\]

**Proof:** Using the expression \( P\{Y > y\} = \int_{y}^{\infty} p(x)dx \), we can rewrite the right-hand side as

\[
\int_{0}^{\infty} P(Y > y)dy = \int_{0}^{\infty} \left( \int_{y}^{\infty} p(x)dx \right)dy
= \int_{0}^{\infty} \left( \int_{0}^{x} dy \right) p(x)dx
= \int_{0}^{\infty} xp(x)dx = E[Y].
\]
Notice that in passing from the first to the second line, we have interchanged the order of integration. (Draw a picture if this calculation is not clear.) This interchange is justified by the fact that the integrand is non-negative.

\[\square\]

**Proposition (Ross, 2.1):** If \(X\) is a continuous random variable with density \(p(x)\) and if \(f: \mathbb{R} \rightarrow \mathbb{R}\), then

\[E[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)dx.\]

**Proof of partial result:** Assume that \(f\) is non-negative. By applying the preceding lemma to the random variable \(Y = f(X)\), we have

\[
E[f(X)] = \int_{0}^{\infty} P\{f(X) > y\}dy
\]

\[
= \int_{0}^{\infty} \left( \int_{x: f(x) > y} p(x)dx \right) dy
\]

\[
= \int_{-\infty}^{\infty} \left( \int_{0}^{f(x)} dy \right) p(x)dx
\]

\[
= \int_{-\infty}^{\infty} f(x)p(x)dx,
\]

where the last equality follows from the fact that \(f(x) \geq 0\) for all \(x \in \mathbb{R}\).

\[\square\]

**Example:** If \(n\) is a non-negative integer, then the \(n\)'th moment of the continuous random variable \(X\) is the expected value of \(X^n\), which can be calculated as:

\[E[X^n] = \int_{-\infty}^{\infty} x^n p(x)dx.\]

**Definition:** Let \(X\) be a continuous RV with density function \(p(x)\). Then the variance of \(X\) is

\[Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2.\]

The equality of the first and second lines can be established using the same calculation applied in the case of a discrete random variable.
2. The Uniform Distribution

**Definition:** A continuous random variable $X$ is said to have the uniform distribution on the interval $(a, b)$ (denoted $U(a, b)$) if the density function of $X$ is

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a, b) \\ 0 & \text{otherwise.} \end{cases}$$

**Interpretation:** To say that $X$ is uniformly distributed on an interval $(a, b)$ means that each point in $(a, b)$ is equally likely to be a value of $X$. In particular, the probability that $X$ lies in any subinterval of $(a, b)$ is proportional to the length of the subinterval: if $(l, r) \subset (a, b)$, then

$$\mathbb{P}\{X \in (l, r)\} = \frac{r - l}{b - a}.$$

**Moments:** Suppose that $X$ has the $U(a, b)$ distribution. Then the $n$’th moment of $X$ is given by

$$E[X^n] = \frac{1}{b-a} \int_a^b x^n dx = \frac{1}{b-a} \frac{1}{n+1} x^{n+1}\bigg|_a^b = \frac{1}{n+1} \left( \frac{b^{n+1} - a^{n+1}}{b-a} \right).$$

It follows that the mean and the variance of $X$ are

$$E[X] = \frac{1}{2}(a+b)$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{1}{12}(b-a)^2.$$

The case $U(0,1)$ is of particular importance. If $X$ is uniform on $(0,1)$, then

$$E[X] = \frac{1}{2} \quad \text{and} \quad Var(X) = \frac{1}{12}.$$

**Remark:** Notice that there is no random variable that is uniformly distributed over an infinite interval such as $(0, \infty)$ or $(-\infty, \infty)$. This follows from the fact that the integral of a constant density over such a region is necessarily infinite.

**Construction:** Suppose that $X_1, X_2, \cdots$ is a sequence of independent Bernoulli random variable, each with parameter $1/2$. If we define

$$X = \sum_{n=1}^{\infty} X_n \left( \frac{1}{2} \right)^n,$$  

then $X$ is a standard uniform random variable. Notice that the random sequence $(X_1, X_2, \cdots)$ is the binary representation of the random number $X$. That $X$ is uniformly distributed can be deduced as follows:
Step 1: We say that a number $x \in \mathbb{R}$ is a dyadic rational number if $x$ has a finite binary expansion, i.e., there exists an integer $n \geq 1$ and numbers $c_1, \cdots, c_n \in \{0, 1\}$ such that

$$x = \sum_{k=1}^{\infty} c_k \left(\frac{1}{2}\right)^k.$$  

Notice that every dyadic rational number is rational, but that not every rational number is dyadic rational, e.g., $x = 1/3$ is not dyadic rational. On the other hand, the set of dyadic rational numbers is dense in $\mathbb{R}$, i.e., for every $z \in \mathbb{R}$ and every $\epsilon > 0$, there exists a dyadic rational number $x$ such that $|z - x| < \epsilon$.

Step 2: Every number $x \in \mathbb{R}$ either has a unique, non-terminating binary expansion, or it has two binary expansions, one ending in an infinite sequence of 0’s and the other ending in an infinite sequence of 1’s. In either case, 

$$\mathbb{P}\{X = x\} = 0.$$  

Step 3: Given numbers $c_1, \cdots, c_n \in \{0, 1\}$, let $x$ be the corresponding dyadic rational defined by the equation shown in Step 1 and let $E(c_1, \cdots, c_n)$ be the interval $(x, x + 2^{-n}]$. Then 

$$\mathbb{P}\{X \in E(c_1, \cdots, c_n)\} = \mathbb{P}\{X_1 = c_1, \cdots, X_n = c_n\} - \mathbb{P}\{X = x\} = \left(\frac{1}{2}\right)^n.$$  

Furthermore, if $0 \leq a < b \leq 1$, where $a$ and $b$ are dyadic rational numbers, then because the interval $(a, b]$ can be written as the disjoint union of finitely many sets of the type $E(c_1, \cdots, c_n)$ for suitable choices of the $c_i$’s, we have that 

$$\mathbb{P}\{X \in (a, b]\} = (b - a).$$  

Step 4: Given any two numbers $0 \leq a < b \leq 1$, not necessarily dyadic rational, we can find a decreasing sequence $(a_n; n \geq 1)$ and an increasing sequence $(b_n; n \geq 1)$ such that $a_n < b_n$, $a_n \to a$ and $b_n \to b$. Then the sets $(a_n, b_n)$ form an increasing sequence, with limit set $(a, b)$, and by the continuity properties of probability measures, we know that 

$$\mathbb{P}\{X \in (a, b]\} = \lim_{n \to \infty} \mathbb{P}\{X \in (a_n, b_n]\} = \lim_{n \to \infty} (b_n - a_n) = b - a.$$  

However, this result implies that $X$ is uniform on $[0, 1]$. 

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