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Citation: Phys. Fluids 22, 126601 (2010); doi: 10.1063/1.3518137
View online: http://dx.doi.org/10.1063/1.3518137
View Table of Contents: http://pof.aip.org/resource/1/PHFLE6/v22/i12
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Lagrangian dynamics in stochastic inertia-gravity waves

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(Received 20 April 2010; accepted 21 October 2010; published online 2 December 2010)

For an idealized inertia-gravity wave, the Stokes drift, defined as the difference in end positions of a fluid parcel as derived in the Lagrangian and Eulerian coordinates, is exactly zero after one wave cycle in a deterministic flow. When stochastic effects are incorporated into the model, nonlinearity in the velocity field changes the statistical properties. Better understanding of the statistics of a passive tracer, such as the mean drift and higher order moments, leads to more accurate predictions of the pattern of Lagrangian mixing in a realistic environment. In this paper, we consider the inertia-gravity wave equation perturbed by white noise and solve the Fokker–Planck equation to study the evolution in time of the probability density function of passive tracers in such a flow. We find that at initial times the tracer distribution closely follows the nonlinear background flow and that nontrivial Stokes drift ensues as a result. Over finite times, we measure chaotic mixing based on the stochastic mean flow and identify nontrivial mixing structures of passive tracers, as compared to their absence in the deterministic flow. At later times, the probability density field spreads out to sample larger regions and the mean Stokes drift approaches an asymptotic value, indicating suppression of Lagrangian mixing at long time scales. © 2010 American Institute of Physics. [doi:10.1063/1.3518137]

I. INTRODUCTION

The study of mixing structure and transport phenomena in a nonlinear, chaotic fluid flow can be traced back to the work of Taylor (1934),1 where he studied the emulsion of a fluid droplet inside another fluid environment subject to background strain and shear. Later, Welander (1955)2 stressed that mixing processes should be understood by the stretching and folding of material elements. Ottino (1989)3 provided a wonderful summary of topics in chaotic mixing. In recent studies, focus has been placed on characterization of flow structures aiming at identifying the topology of a nonlinear, chaotic flow. In chronic orders, Eulerian coherent structures [Okubo (1970), Chong et al. (1990), Weiss (1991), Jeong and Hussain (1995)]4–7 were first developed to extract flow regions that enhance mixing (regions of high strain) and those that inhibit mixing (regions of high vorticity). Haller (2000, 2001, 2005)8–10 and Haller and coauthors (2000, 2003)11,12 developed the theory of Lagrangian coherent structures (LCS), capable of the objective extraction of mixing structures (hyperbolic, parabolic, and elliptic regions) in time aperiodic, chaotic flows. Most importantly, LCS identifies the distinguished material lines-surfaces that are most conducive to enhancement or inhibition of the mixing of tracers, and thus they are exactly the topological structures that Welander (1955)2 emphasized.

In most of the studies on LCS to date, the focus has been on their extraction in deterministic flows. There the chaotic trajectories follow a determined fashion based on the background flow and hence passive tracers will follow the dynamics organized by the LCS, provided that the evolution of the flow structures is much slower than the evolution of tracer trajectories [Haller (2001)].9 Geophysical flow is one example where the LCS theory is valid. As such, Sapsis and Haller (2009)13 and Tang et al. (2010)14 discussed the application of LCS in identification of mixing structures in hurricane Isabelle and a subtropical jet stream near Hawaii. However, with geophysical applications in mind (as with many other applications), the velocity information is not well resolved due to constraints on the data acquisition (measurements or simulations) and thus there are inherent subgrid scale uncertainties that affect the dynamics of a tracer. As such, Olcay et al. (2010)15 studied the influence of random noise on LCS for a vortex ring field through a Lagrangian particle tracking method, assuming Gaussian white noise.

In this paper, we seek to obtain the stochastic mixing structure of a nonlinear background flow subject to anisotropic white noise. In addition to computing the Lagrangian trajectories by the ensemble mean of a cluster of particles, we track the evolution of the probability density field through the Fokker–Planck (FP) equations [Sobczyk (1991)].16 With its solution we will be able to construct mean trajectories and higher order moments, hence better characterize the influence of stochasticity on Lagrangian mixing.

We focus on studying the advection of passive tracers in an inertia-gravity wave (IGW) field. There are several motivations of the flow field under consideration. On the physical end, IGWs are ubiquitous in the environment [Garzoli and Katz (1981), Eckermann and Vincent (1993), Plougonven et al. (2003), Lane et al. (2004)].17–20 Their generation, advection, nonlinear interaction, and dissipation are associated with large energy and momentum transfer and these dynamical processes play important roles in the global energy budget of the atmosphere and ocean circulations. At smaller scales, the breaking of IGW in the upper-troposphere and lower stratosphere (UTLS) is known to form clear-air turbulence, a primary source of aviation hazard. The study of coherent motion of tracer dynamics in such fields can thus out-
line regions of instability, likely candidates for aviation hazards [Tang et al. (2010)]. The study of Lagrangian transport processes in this region also leads to better understandings of the structures of isentropic and vertical mixing [Legras and d’Ovidio (2007)]. Indeed, dynamics, chemistry, microphysics, and radiation are fundamentally interconnected in the UTLS region [Mahalov and Moustaoi (2009, 2010)]. As such, characterization of scalar mixing and transport via random processes in this region are very important for development of improved models. On the mathematical end, IGW is a nice prototypical model for the study of random noise on Lagrangian dynamics, as the mean motion over a wave cycle, the Stokes’ drift [Craik (2005), Stokes (1847)]; is exactly zero when no noise is present. Therefore, any net effect as derived from the study of the FP equations would directly impact the issue of stochasticity. Through studies of the FP equations we will be able to characterize the mean and higher order moments of this random process.

The stochastic effects on Stokes drift of Lagrangian tracer particles have been first studied by Jansons and Lythe (1998). They examined the dynamics of Lagrangian tracer particles subject to Gaussian white noise in 1D multichromatic wave flows and find analytical expressions for the Stokes drift subject to the random noises. Following their studies, Jansons (2007) studied the stochastic Stokes drift for inertial particles. For geophysical flows, wave-generated transport associated with stochasticity was studied in Restrepo and Leaf (2002), where the drift velocities for progressive and standing 2D waves were obtained by numerical simulations. Stochastic Lagrangian drifts have also been incorporated in wave driven circulation models in Restrepo (2007) to parametrize wave breaking effects. In this study, our focus is both on obtaining the stochastic Stokes drift on a monochromatic IGW so as to obtain the Lagrangian mixing topology associated with stochasticity and quantifying the statistics of the various moments for this random process.

Being able to characterize the non-Gaussianity (intermittency) in nonlinear processes is important in many disciplines, including studies on Lagrangian tracer dynamics. It provides quantitative information of how a process is different from zero-mean, symmetric, Gaussian processes. This knowledge is important in improving both the modeling of tracer mixing and the detection of tracer transport in nonlinear flow fields. Such characterizations are based on the study of probability density functions. A general observation from experimental and numerical data is the broadening of the tail of a Gaussian. Numerous studies have been dedicated into the characterization of this broadening effect. To name a few of these studies among a vast literature, in Bronski and McLaughlin (2000), asymptotic behaviors of large moments were obtained analytically for a random linear shear model. Bourlioux and Majda (2002) evaluated intermittency of passive scalars associated with a mean gradient and find four asymptotic regimes of intermittency based on choices of the Péclet number and the flow forcing period. Kramer et al. (2003) carried out a comparative study on closure approximations for passive scalar intermittency in a class of shear models. Sukhatme (2004) analyzed the regime that probability density functions exhibit strange eigen-modes (attain self-similarity in finite-time). Tartakovsky et al. (2009) analyzed intermittency in reacting flows arising from uncertainties in reaction rates. For atmospheric flows, in the stratosphere, the probability density for concentration and tracer gradients have been studied in Hu and Pierrehuembert (2001, 2002).

In this paper, to study the evolution and the mixing patterns of a passive tracer, we first solve the FP equation over one wave cycle to obtain the deviation from the deterministic solution. Mean drifts are obtained from the expectation of displacements. Indeed, we start the simulation with different initial conditions to measure chaotic mixing of different tracers as induced by mean drifts. LCS (absent in the deterministic flow) is obtained through the computation of finite-time Lyapunov exponents (FTLE) [Haller (2001)] on the stochastic mean trajectories. The probability density fields are characterized by discrete moments, including variance, skewness, and kurtosis. In addition to the dynamics over a single wave cycle, we run a suite of simulations with different variances to longer times when the expectations reach some asymptotic value. This allows us to study the long-time behavior of a tracer and the dependence of statistics on different variances. Our solutions from the FP equation are tested against Lagrangian particle tracking methods to ensure their fidelity. In fact, it is possible to obtain the analytical expressions for different moments because we have a closed system due to the form of the IGW. We only show the analyses for the first and second moments in this paper and compute higher order moments numerically using the probability density calculated from the FP equation.

The rest of the paper is organized as follows. In Sec. II we introduce the nondimensional FP equation proper to our problem. In Sec. III we discuss some analytic results for the mean trajectories. In Sec. IV we discuss numerical results from computation of the FP equations and the Lagrangian particle tracking methods. In Sec. V we draw conclusions and discuss future directions. The detailed derivations of analytical expressions are given in the Appendix.

II. MATHEMATICAL FORMULATION

The linear solution of an IGW is given by a polarized velocity field [Gill (1982)]

\[
\mathbf{u} = (u,v,w) = \left( u_0 \cos \phi, u_0 \frac{f}{\omega} \sin \phi, -u_0 \frac{k}{m} \cos \phi \right),
\]

where \(u_0\) is a velocity scale, \(k\) and \(m\) are the horizontal and vertical wavenumbers, \(f\) is the Coriolis frequency, \(\omega\) is the wave frequency, and \(\phi = kx + mz - \omega t\) is the wave phase. Realistic values of these parameters for atmospheric flows are given later in the text. For a deterministic flow, the Lagrangian particle trajectories can be integrated from Eq. (1) [Stokes (1848), Lighthill (1979)] and it can be shown that the Lagrangian trajectory is exactly the same as the time integral of the Eulerian velocity at all times, due to the exact cancellation of \(kx\) and \(mz\) in \(\phi\). Therefore, no Lagrangian mixing of tracers occurs after each wave period for the deterministic flow.
In this paper, we consider perturbations to Eq. (1) by anisotropic, homogeneous Gaussian white noise with constant variances $\sigma_h^2, \sigma_v^2$ along the horizontal and vertical axes. These are related to eddy diffusivities as $\sigma_h^2 = 2 \epsilon_h$, $\sigma_v^2 = 2 \epsilon_v$. The choice of anisotropy is motivated by turbulent motion in density stratified flows driven by shear. Specifically, in a density stratified environment, vertical fluctuations are suppressed due to the energy required to overcome stable stratiﬁcation. However, it is pointed out in Sarkar (2003)\(^{39}\) that, although suppressed by stratiﬁcation, vertical motion is still coupled with the horizontal components of the ﬂuctuations, and thus we still consider the full three-dimensional ﬂow ﬁeld, with the scales of ﬂuctuations imposed by the appropriate choices of variances. Of course, this particular paper does not fully capture the nonstationary, nonhomogeneous nature of small-scale turbulent motions. Our goal is to characterize the heterogeneity of tracer mixing that is caused by randomness in the nonlinear velocity data. As will be demonstrated in the following sections, even the simplest stochastic process such as Gaussian white noise can induce such heterogeneity. In addition, with this simple noise structure we can derive analytic solutions for the mean trajectories, which can be used to conﬁrm the accuracy of the numerical simulations.

Suppose that the stochastic trajectory of a tracer $x_t=(x_t, y_t, z_t)$ satisﬁes the following system of stochastic diﬀerential equations:

$$\begin{align*}
\text{d}x_t &= u dt + \sigma_h \text{d}W_t^{(1)}, \\
\text{d}y_t &= v dt + \sigma_h \text{d}W_t^{(2)}, \\
\text{d}z_t &= w dt + \sigma_h \text{d}W_t^{(3)},
\end{align*}$$

where $(u, v, w)$ are given in Eq. (1) and $\mathbf{W}=(W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$ is a standard vector Wiener process $\mathbf{W}_t$ whose components are independent from each other. The joint probability density $P$ of the stochastic velocity ﬁeld is a solution to the FP equation, given as [Sobczyk (1991)]\(^{16}\)

$$P_t + \mathbf{u} \cdot \nabla P = \frac{1}{2} \sigma_h^2 \nabla_x^2 P + \frac{1}{2} \sigma_v^2 P_{zz}.$$  

Using the characteristic horizontal length scale $L_h$ and the time scale of a wave period $T=2\pi/\omega$, the nondimensional FP equation is

$$P_+ + U(\cos \Phi P_x + \sin \Phi P_y - \cos \Phi P_z) = \frac{1}{2} D_h \nabla \cdot \mathbf{P}_t + \frac{1}{2} D_v P_{zz},$$

where $\tau=t/T$ is the nondimensional time, $U=u_0 T/L_h$ is the nondimensional velocity scale, $\Phi=2\pi(X+Z-\tau)$ is the nondimensional wave phase, $(X,Y,Z)=(x,y,Rz)/L_h$ is the nondimensional coordinates, $R=m/k$ is the aspect ratio, and $(D_h,D_v)=(\sigma_h^2, R^2 \sigma_v^2)/L_h^3$ is the nondimensional variances in the horizontal and vertical directions, respectively. With the simple noise structure, Eq. (4) is an advection-diﬀusion equation for the IGW.

The nonlinear velocity ﬁeld makes the probability density $P$ in Eq. (4) analytically intractable for positive values of the variances. However, some averaged quantities, such as the mean and variances, can be obtained through the hierarchy of moments outlined in Young et al. (1982),\(^{40}\) with the deﬁnition of an advected coordinate rotating with the ﬂow. In their paper, the advection-diﬀusion of a polarized velocity ﬁeld rotating in the horizontal plane was considered and the problem was solved in the context of extracting effective diffusivity through shear dispersion. The diﬀerence among wave phases was deliberately ﬁltered out through the choice of a line source. One of our aims in this study is indeed examining the nontrivial mixing pattern arising from diﬀerent phases of the IGW. As such, our initial conditions are highly localized and the polarized ﬂow is tilted from Young et al. (1982).\(^{40}\) In consideration of higher order moments, we note that the method of moments becomes progressively more diﬃcult as the order of the moment increases. Hence, Eq. (4) is also solved numerically to obtain a more complete picture of the statistics. Mean trajectories obtained from $P$ are compared to analytic solutions obtained from an explicit solution to Eq. (2) to ensure accuracy of the numerical solutions.

As mentioned above, since the nontrivial stirring pattern of an IGW coupled with stochastically may give rise to a nontrivial mixing pattern of tracers, we examine this mixing through the mean trajectories. Olcay et al. (2010)\(^{15}\) studied the inﬂuence of random noise on Lagrangian mixing, where stochasticity is introduced by adding Gaussian white noise to the background velocity ﬁeld for a set of initial conditions starting at the same location and time (Lagrangian particle tracking method). The ensemble of trajectories is considered in the computation of FTLE. To the best of our knowledge, this paper is the ﬁrst to evaluate LCS due to random noise for 3D ﬂows subject to anisotropic perturbation and with consideration of higher moments. Here, the FTLE are evaluated from the mean trajectories after one wave cycle. Speciﬁcally, we compute

$$\mathbf{M}(\mathbf{X}_0) = \left( \frac{\partial \mathbb{E} [\mathbf{X} (T; \mathbf{X}_0)]}{\partial \mathbf{X}_0} \right)^T \left( \frac{\partial \mathbb{E} [\mathbf{X} (T; \mathbf{X}_0)]}{\partial \mathbf{X}_0} \right),$$

$$\text{FTLE}(\mathbf{X}_0) = \frac{1}{2T} \log \lambda_{\text{max}}(\mathbf{M}),$$

where $\mathbf{M}$ is the Cauchy–Green strain tensor, $\mathbf{X}_0$ is the initial location at $\tau=0$, $\mathbb{E}[\mathbf{X}] = \int \int \int \mathbf{P} \mathrm{d}x \mathrm{d}y \mathrm{d}z$ is the expectation of the initial condition after one wave period, $(\cdots)^T$ denotes the transpose of the deformation matrix, and $\lambda_{\text{max}}$ evaluates the largest eigenvalue of $\mathbf{M}$ [Haller (2001)].\(^{9}\) The FTLE
field highlights regions in physical space where intense Lagrangian mixing occurs during the finite-time period of evaluation.

III. ANALYTIC SOLUTIONS

We present in this section the analytic solutions to the first and second order moments of the governing system. The details of the derivation can be found in the Appendix. Rewriting Eq. (2) in nondimensional coordinates, we have

\[ \dot{X}_1 = U \cos \Phi + D_h^{1/2} W_1^{(1)}, \quad \dot{Y}_1 = U \sin \Phi + D_h^{1/2} W_1^{(2)}, \]

\[ \dot{Z}_1 = -U \cos \Phi + D_v^{1/2} W_1^{(3)}, \]

where \( \dot{q} = dq/d\tau \) for some quantity \( q \), \( U \) is the nondimensional velocity scale, \( \Phi = 2\pi(X + Z - \tau) \) is the wave phase, and \( X, Y, Z \) are the nondimensional coordinates.

Considering a tracer with some initial distribution at time \( \tau = 0 \), we find that the mean position of a tracer at time \( \tau \) is

\[ \bar{X} = E[X(\tau)] = E_X + \frac{UE_2 \pi D - \exp(-2\pi^2 D\tau)(\pi D \cos 2\pi\tau - \sin 2\pi\tau)}{1 + \pi^2 D^2}, \]

\[ \bar{Y} = E[Y(\tau)] = E_Y - \frac{UE_2 \pi D - \exp(-2\pi^2 D\tau)(\cos 2\pi\tau + \pi D \sin 2\pi\tau)}{1 + \pi^2 D^2}, \]

\[ \bar{Z} = E[Z(\tau)] = E_Z - \frac{UE_2 \pi D - \exp(-2\pi^2 D\tau)(\pi D \cos 2\pi\tau - \sin 2\pi\tau)}{1 + \pi^2 D^2}. \]

For comparison, we observe that the particle trajectory starting from \( X_0 = 0 \) in the nondimensional deterministic velocity field is

\[ X(\tau) = (-U \sin \Phi/2\pi, U \cos \Phi/2\pi) \]

\[ -U/2\pi, U \sin \Phi/2\pi). \]

For very small variances \( D_h, D_v < 1 \) and over small times, the mean trajectory remains close to the deterministic trajectory Eq. (9). Furthermore, as \( \tau \to \infty \), we observe that the mean trajectory approaches \((0, -U/2\pi, 0)\), the center of the ellipse where the deterministic trajectory resides. This is because the phase \( \Phi, \) defined as

\[ \Phi = 2\pi(X_0 + Z_0 - \tau) + 2\pi D_h^{1/2} W_1^{(1)} + 2\pi D_v^{1/2} W_1^{(3)}, \]

where \( W_i = (D_h/D)^{1/2} W_i^{(1)} + (D_v/D)^{1/2} W_i^{(3)} \), evolves as a Brownian motion with a constant negative drift, which is the advected coordinate. As time progresses, the distribution of the phase \( \Phi \text{ mod } 2\pi \) tends to the uniform distribution on \([0, 2\pi]\), so that the mean position of an ensemble of independent particles governed by the stochastic vector field is asymptotic to the time average of the position of a deterministic particle over the course of a single cycle. On the other hand, in the limit of large variance \( D_h, D_v \gg 1 \), the mean trajectories remain near the origin, indicating that the stochastic process behaves like a Brownian motion and is only weakly influenced by the nonlinear background flow. For intermediate values of \( D_h, D_v \), the mean tracer trajectory will approach \([U D/2(1 + \pi^2 D^2), -U/2\pi(1 + \pi^2 D^2), -U D/2(1 + \pi^2 D^2)]\).

Analytical expressions for the second order moments are
\[
\text{Var}(X) = E_{X^2} - \bar{X}^2 + D_\tau \tau + U9 \left\{ \frac{2E_{X\phi}}{\mu} (e^{\mu \tau} - 1) - \frac{U \tau}{\mu'} + \frac{U^2}{\mu^2} (e^{\mu \tau} - 1) + \frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} + 4 \pi i D_\tau E_{\phi} \left( \frac{\tau e^{\mu \tau}}{\mu' - \mu} + \frac{1}{\mu^2} \right),
\]
\[
\text{Var}(Y) = E_{Y^2} - \bar{Y}^2 + D_\tau \tau + U9 \left\{ \frac{2E_{Y\phi}}{\mu} (e^{\mu \tau} - 1) - \frac{U \tau}{\mu'} + \frac{U^2}{\mu^2} (e^{\mu \tau} - 1) - \frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) \right\},
\]
\[
\text{Var}(Z) = E_{Z^2} - \bar{Z}^2 + D_\tau \tau + U9 \left\{ -\frac{2E_{Z\phi}}{\mu} (e^{\mu \tau} - 1) - \frac{U \tau}{\mu'} + \frac{U^2}{\mu^2} (e^{\mu \tau} - 1) + \frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} - 4 \pi i D_\tau E_{\phi} \left( \frac{\tau e^{\mu \tau}}{\mu' - \mu} + \frac{1}{\mu^2} \right),
\]
\[
\text{Cov}(X,Y) = E_{XY} - \bar{X} \bar{Y} + U9 \left\{ \frac{E_{Y\phi}}{\mu} (e^{\mu \tau} - 1) \right\} + U9 \left\{ \frac{E_{X\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} + 2 \pi i D_\tau E_{\phi} \left( \frac{\tau e^{\mu \tau}}{\mu' - \mu} + \frac{1}{\mu^2} \right),
\]
\[
\text{Cov}(X,Z) = E_{XZ} - \bar{X} \bar{Z} + U9 \left\{ \frac{E_{Z\phi}}{\mu} - \frac{E_{X\phi}}{\mu'} (e^{\mu \tau} - 1) \right\}
+ U9 \left\{ \frac{U \tau}{\mu} - \frac{U}{\mu^2} (e^{\mu \tau} - 1) - \frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} + [2 \pi i (D_\tau - D_\rho) E_{\phi}] \left\{ \frac{\tau e^{\mu \tau}}{\mu' - \mu} + \frac{1}{\mu^2} \right\},
\]
\[
\text{Cov}(Y,Z) = E_{YZ} - \bar{Y} \bar{Z} + U9 \left\{ \frac{E_{Z\phi}}{\mu} - \frac{E_{Y\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} - U9 \left\{ \frac{E_{Y\phi}}{\mu'} (e^{\mu \tau} - 1) \right\}
+ U9 \left\{ -\frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) - \frac{UE_{2\phi}}{\mu'} (e^{\mu \tau} - 1) \right\} + 2 \pi i D_\tau E_{\phi} \left( \frac{\tau e^{\mu \tau}}{\mu' - \mu} + \frac{1}{\mu^2} \right),
\]

where \( \mu = -(2 \pi^2 D + 2 \pi) \), \( \mu' = -(8 \pi^2 D + 4 \pi) \), \( E_{X\phi} = E_1 + i E_2 \)
\( = E[\sin \Phi_0] + i E[\cos \Phi_0] \), \( E_{Z\phi} = E[\sin 2\Phi_0] + i E[\cos 2\Phi_0] \), and \( E_{X^2}, E_{Y^2}, E_{XZ}, E_{Z\phi}, E_{\phi^2}, E_{Y\phi} \), and \( E_{2\phi} \) are the expectations of \( X^2, Y^2, Z^2, X, Y, Z, X^2, Y^2, \) \( Z^2 \) and \( E_{2\phi} \) at time \( \tau = 0 \), respectively.

As seen, over long times, the variances scale linearly with \( \tau \) and are augmented by \( U^2 D \tau / 2 (\pi^2 D^2 + 1) \) due to the nonlinear background flow. Considering eddy diffusivities in geophysical flows, where vertical diffusion is strongly limited by density stratification \( (D_\rho \ll D_\tau) \), the above expression implies that \( \text{Var}(Z) \) remains finite due to horizontal diffusion. In this limit, the covariances \( \text{Cov}(X,Y) \) and \( \text{Cov}(Y,Z) \) also stay finite while \( \text{Cov}(X,Z) \) decreases linearly with \( \tau \) due to the symmetry in \( X \) and \( Z \).

IV. NUMERICAL RESULTS

The probability density \( P \) naturally satisfies vanishing boundary conditions at infinity. To ensure that the problem is numerically tractable with high precision, we solve Eq. (4) in a finite domain subject to periodic boundary conditions and require \( P \) to be negligibly small near the computational boundaries compared to its maximum value inside the domain for all time (the ratio between them is maintained at \( 10^{-8} \) for all time). The numerical solver we use is described in detail in Bewley (2011). Since periodic boundary conditions are applied in all directions, derivatives are treated with a pseudospectral method. The low storage third-order Runge–Kutta–Wray method was used for time stepping and diffusive terms are treated implicitly with the Crank–Nicolson method. In order to prevent spurious aliasing due to nonlinear interactions between wavenumbers, the largest 1/3 of the horizontal wavenumbers are truncated using the 2/3 dealing rule (Orszag (1971)). The initial probability distribution was taken to be Gaussian with density \( P \) satisfying \( \int P dX dY dZ = 1 \) and the nondimensional variances of \( P \)
were \( \sigma_X^2 = \sigma_Y^2 = 0.005 \) and \( \sigma_Z^2 = 0.005 \). The differential equation is then integrated in time over one wave period \( T \) and longer to obtain the statistics of the randomness of the wave field. We use variance, skewness, and excess kurtosis to measure the statistics of the probability density. For Brownian motion, the variance grows linearly with \( \tau \). Skewness measures the asymmetry of the probability density and is zero for a per-
fectly symmetric field. Excess kurtosis measures the flatness of the probability density. It is zero for the Gaussian and −3 for a uniform probability density. Any deviation from these standard values indicates deviation from a Gaussian process. In addition, we compute the mean trajectories initiated from different initial conditions to characterize the Lagrangian mixing structure. Note that due to the periodic nature of the wave field, the transition density \( P \) can be obtained for any initial location by simply varying the initial phase \( \Phi_0 \). This is used when constructing the Lyapunov exponents of the mean trajectories.

As discussed in Sec. I, we consider an IGW generated by the tropospheric midlatitude jet stream [Plougonven et al. (2003)]\(^{19} \) In this setting, a typical velocity scale is \( u_0 = 7 \) m/s, wave period \( T = 12.2 \) hr, horizontal length scale \( L_0 = 220 \) km, and aspect ratio \( R = 100 \). Stochasticity is introduced through eddy diffusivities estimated for the UTLS. The vertical eddy diffusivity near the lower stratosphere, as discussed in Wilson (2004),\(^{43} \) varies from \( O(0.01) \) to \( O(1) \) m\(^2\)/s. For the primary case I, we use \( \epsilon_z = 0.1 \) m\(^2\)/s, in line with observations outside the polar vortex [Legras et al. (2003)].\(^{44} \) The horizontal eddy diffusivity has stronger variability and we use the value \( \epsilon_h = 1000 \) m\(^2\)/s. In addition to case I, we examine the effects of different diffusivities on the mean trajectories by simulating the probability density for three progressively larger diffusivities, while holding the ratio between the horizontal and vertical diffusivities constant. The choices of these additional cases are still in physically realizable ranges. These additional simulations reveal how different diffusivities can affect the statistics of tracer dynamics in IGW.

### A. Statistics after one wave cycle

We assume that the initial Gaussian probability density is centered at the middle of the computational box and solve Eq. (4) over one wave period. The temporal evolution of the statistics of an initial condition at wave phase \( \Phi_0 = 0 \) is shown in Fig. 1. Unless indicated otherwise, solid curves denote measures in \( X \), dashed curves denote measures in \( Y \), and dashed-dotted curves denote measures in \( Z \). We note that for some initial probability distributions, the nonlinear background flow is capable of advecting the probability density around and driving it away from a Gaussian distribution during a wave cycle, even without stochasticity. The analytical solution for the advected probability density, started from an isotropic Gaussian distribution, is

\[
P(X, \tau) = \frac{1}{(2\pi)^{3/2}\sigma_0} \exp\left(-\frac{|\tilde{X}|^2}{2\sigma_0^2}\right),
\]

where

\[
\tilde{X} = X + (U \sin \Phi/2\pi, -U \cos \Phi/2\pi - U/2\pi,\]

\[
- U \sin \Phi/2\pi)
\]

is the advected coordinate that corresponds to the initial location \( X_0 \) of a tracer which is at location \( X \) at time \( \tau \) [Young et al. (1982)]\(^{40} \) and \(|\tilde{X}|^2\) denotes the Euclidean norm of the advected coordinate. This expression is valid because the initial standard deviation is isotropic. For comparison, we use thick curves to indicate the simulation results from case I and thin curves to indicate the deterministic case. As seen in Fig. 1(a), the time dependent standard deviation \( \sigma \) for \( X \) and \( Z \) is at its maximum at half period \( T/2 \), whereas the time dependent standard deviation for \( Y \) has maxima around \( T/4 \) and \( 3T/4 \). This is not surprising, since for two trajectories initiated near \( \Phi_0 = 0 \), cosine function creates the largest stretching half way through a period, whereas the sine function reaches extremes at \( T/4 \) and \( 3T/4 \). For the stochastic case, the observations that the second peak in \( \sigma_Y \) is larger than the first peak and that the standard deviations return to values larger than their initial conditions are reassuring since stochastic processes should work to increase variance. This trend is confirmed from comparison between the thick and thin curves.

The temporal evolution of the skewness \( \mathcal{S} \) and the excess kurtosis \( \mathcal{K} \) also show that there is distortion away from a Gaussian distribution. To be exact, the case with zero stochasticity indicates a distortion of the Gaussian structure due to the nonlinear background flow and this non-Gaussian structure evolves over a wave cycle to return to the initial Gaussian structure instantaneously. Stochasticity works to reduce the non-Gaussian behaviors of \( \mathcal{S} \) and \( \mathcal{K} \) created by the nonlinear background flow, even though the statistics show that the distribution is not Gaussian at the moment of the

---

**FIG. 1.** (a) Temporal evolution of variance \( \sigma \) of the probability density \( P \) for initial phase \( \Phi_0 = 0 \). The thick version of solid, dashed, and dashed-dotted curves denote the standard deviations in the \( X \), \( Y \), and \( Z \) directions, respectively. The thin version of these curves shows respective variances computed from a case started from the same initial conditions but with no diffusion. (b) Temporal evolution of the skewness \( \mathcal{S} \). (c) Temporal evolution of excess kurtosis \( \mathcal{K} \). The line styles of (b) and (c) are the same as (a).
completion of a wave cycle. As we show later, stochasticity prevails in the long run in the determination of the hierarchy of moments, and all statistics indicate a Gaussian process whose skewness may be affected by the initial conditions.

Next we consider Lagrangian stirring induced by nearby tracers. Because of the spatial asymmetry, we do not expect the mean trajectories to return to their initial locations. As such, nontrivial Lagrangian mixing will occur as compared to the deterministic case with no Lagrangian mixing. We compute the mean trajectories $E(X)$, $E(Y)$, and $E(Z)$ starting from different initial conditions and integrate over one wave cycle. We then use FTLE discussed in Sec. II as the measure of chaotic mixing to characterize the stochastic stirring of fluid particles. From Eq. (7) we expect the end locations of mean trajectories for different initial phases after one wave cycle to behave as trigonometric functions, since the evolution of the mean trajectory at $\tau = 1$ is only a function of $\Phi_0$, implicitly embedded in $E_X$, $E_Y$, $E_Z$, $E_1$, and $E_2$. As shown in Fig. 2(a), for case I, the mean trajectory can be described as

$$E[X] = -A \sin \Phi_0, \quad E[Y] = -A \cos \Phi_0,$$

$$E[Z] = A \sin \Phi_0,$$

where $A \approx 0.013$ and $\Phi_0$ is the initial phase a tracer assumes at time 0. This significantly simplifies the computation of the FTLE, since the largest eigenvalue of $M$ takes the form

$$\lambda = 1 + A^2(1 + \cos \Phi_0^2) + \sqrt{A^4(1 + \cos \Phi_0^2)^2 + 2A^2(1 + \cos \Phi_0^2)}.$$

Starting from a uniform set of grid points, the initial conditions are deformed by the flow and we can locate the material surfaces that attract or repel nearby trajectories. We show the deformation of these initial conditions after one wave cycle in Fig. 2(b). The repellers are highlighted by the three solid lines. Note that we have exaggerated the values of mean trajectories by a factor of 5 to make the repellers visible. Since the FTLE field is independent of $Y$, we only show the $X-Z$ section of this scalar field in Fig. 2(c). Enhanced Lagrangian mixing is found along the half-integer phase line (dark shaded regions, color contour online), which deviates from the trivial mixing case of a deterministic flow.

It is worth mentioning here that even though we have periodicity in our system and thus can extract the infinite time Lyapunov exponent, we still only focus on its finite-time counterpart. The reason is that for the spatial structure of Lagrangian mixing, it is the geometry, rather than the exact value of Lyapunov exponent, that plays important roles. Because of the periodic behavior of the mean trajectories, the geometry we obtain from finite-time is the same as that computed from infinite time. However, for physically relevant tracers, such as ozone, their chemical properties will eventually become important over long time scales, hence the infinite time Lyapunov exponent will not correctly characterize behaviors over infinite times.

### B. Different variances and long-term behavior

In order to evaluate the effects of different variances on the mean trajectories, we ran four cases initialized at phase $\Phi_0 = 0$ with the variances listed in Table I. The case with zero variance is included for reference. The results of these simulations are summarized in Fig. 3. The thick solid circle in Fig. 3 shows the mean trajectory associated with the deterministic flow with no stochasticity, as described in Eq. (12). The mean trajectories over one wave cycle for different variances are shown as dashed-dotted curves inside the circle, with their end positions marked by the dots. As expected, the mean trajectories move away from the big circle as variance is increased. Indeed, because of the different diffusion time scales associated with the different variances, comparison between the trajectories at one fixed time period is less meaningful than the asymptotics. (This, however, does not invalidate the evaluation of FTLE at one wave period, as

<table>
<thead>
<tr>
<th>Case</th>
<th>$\sigma_{x}^{2}$ (m$^2$/s)</th>
<th>$\sigma_{y}^{2}$ (m$^2$/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>I</td>
<td>2000</td>
<td>0.2</td>
</tr>
<tr>
<td>II</td>
<td>6324</td>
<td>0.6324</td>
</tr>
<tr>
<td>III</td>
<td>20000</td>
<td>2</td>
</tr>
<tr>
<td>IV</td>
<td>63240</td>
<td>6.324</td>
</tr>
</tbody>
</table>
Lagrangian mixing is already taking place within finite time.) As such, we continue the simulation until the mean trajectories settle to (approximately) a single point and plot them as the crosses inside the circle. For clarity and reference, we continue to plot the mean trajectory for case IV with the largest variance. The asymptotic position of this trajectory is the furthest from the center of the circle among the four cases considered. As we move toward smaller variance, the mean trajectory settles toward the center of the circle. Of course, when the variance is very large, the asymptotic point will coincide with the starting point, as it diffuses too rapidly and does not feel the nonlinear background flow field. For comparison, we also plot in Fig. 3(a) the asymptotic positions computed analytically from Eq. (7) as the dashed-dotted curve, assuming the same initial probability as in the simulation. Here, $E_x=E_y=E_Z=E_1=0$, $E_z=\exp[-2\pi^2(\sigma_x^2+\sigma_y^2)]\exp(-0.02\pi^2)$. Thus, mean trajectories in the simulation starting from $E[X(0)]=0$ will asymptote to position $UE_x(\pi D, -1, -\pi D)/2(\pi D^2+\pi)=U\exp(-0.02\pi^2)(\pi D, -1, -\pi D)/2(\pi D^2+\pi)$. It is apparent that the crosses from the simulation fall exactly onto the dashed-dotted curve, as expected.

We are interested in learning other statistics of the tracer dynamics over long-time. In Sec. III we obtain analytical expressions for second order moments. These analytical expressions are compared against the numerical simulations in Figs. 3(b) and 3(c). Using the initial Gaussian profile, we find that $E_{\phi}=\exp(-0.02\pi^2)$, $E_{2\phi}=\exp(-0.08\pi^2)$, $E_{XX}=0.01\pi\exp(-0.02\pi^2)$, $E_{YY}=0$, and $E_{ZZ}=0.01\pi\exp(-0.02\pi^2)$. In Fig. 3(b), we show the comparison between variances $\text{Var}(X)$ (solid), $\text{Var}(Y)$ (dashed), and $\text{Var}(Z)$ (dashed-dotted). We also plot in Fig. 3(c) the covariances $\text{Cov}(X,Y)$ (solid), $\text{Cov}(X,Z)$ (dashed), and $\text{Cov}(Y,Z)$ (dashed-dotted). The thick curves are from numerical simulations up to $\tau=11$. Analytical expressions are computed up to $\tau=12$ and shown in thin curves. Clearly, the comparison shows that both results are identical. Specifically, to the leading order, the variances scale linearly with $\tau$. $\text{Cov}(X,Z)$ also scale linearly with $\tau$ due to the $X,Z$ symmetry whereas $\text{Cov}(X,Y)$ and $\text{Cov}(Y,Z)$ asymptote to a constant.

In Fig. 4 we show the standard deviation, skewness, and excess kurtosis over time for case III. Figure 4(a) shows a log-log plot of the standard deviations with a straight line.
indicating a scaling of $t^{1/2}$. At the end of case III (and also for cases I and II, not shown), the standard deviations approach a scaling slightly smaller than $t^{1/2}$. In contrast, as we show in the next subsection, in case IV the standard deviations do indeed approach a scaling of $t^{1/2}$ at the end. This observation suggests that in the first three cases the simulations probably have not been run long enough for $\sigma$ to closely approach the infinite time scaling. Upon examination of other discrete moments, we find that the excess kurtosis in Fig. 4(c) returns to Gaussian with $K$ returning to 0. However, the skewness in $X$ and $Z$, shown in Fig. 4(b), asymptotes to a nontrivial value that depends on the initial phase. The isocontour clearly indicates a nontrivial skewness in the $X$-direction, which is not removed by stochasticity.

Nonzero values of the skewness can also occur if the process is at its intermediate time scale. To exclude this possibility from the simulation results we use $P$ to calculate general moments [Ferrari et al. (2001)]

$$\langle |q|^s \rangle = \int_V (q - E(q))^s P dV$$

(16)

for quantity $q$ and approximate the power law $\langle |X|^t \rangle \sim t^{s/2}$ at large times. We find that $\gamma_s = s/2$ for $s \in [0, 10]$, consistent with a strong self-similar, normal diffusion process. Hence the convergence of the skewness to nonzero values indicates that asymmetry arises from shear dispersion in IGW.

C. Lagrangian particle tracking methods

In addition to comparison with analytic solutions, we also check our computation of the FP equation by comparing with results from Lagrangian particle tracking methods [Crimaldi et al. (2008)]. For each individual case we seed $10^5$ initial conditions that take the same initial distribution as the computation of the FP equation and obtain various statistics. The initial conditions are iterated forward in time subject to the equation

$$X_i = X_{i-1} + U(X_{i-1}) \Delta \tau_i + Z \sqrt{D_k \Delta \tau_i},$$

(17)

where $X_i$ is the $i$th iteration of a particle trajectory, $U$ is the nondimensional velocity field, $Z$ is a three-dimensional Gaussian process with zero-mean and unit variance, $D_1 = D_2 = D_3$, $D_1 = D_2 = D_3$, are the nondimensional variances, and $\Delta \tau_i$ is the time stepping.

The ensemble of trajectories $E(X) = N^{-1} \sum_{i=1}^{N} X_j$, where $N = 10^5$ is the number of samples, are used to compute the mean drift due to stochastic noise. The standard deviation is computed as $\sigma(X) = \sqrt{N^{-1} \sum_{i=1}^{N} (X_j - E(X))^2}$ to compare with the solution from Eq. (4). We have computed these for all cases of diffusivities but for clarity we only show the results for the case with the highest variance.

Figure 5(a) shows the comparison between mean trajectories computed from the FP Eq. (4) (solid curve) and from the Lagrangian particle tracking method Eq. (17) (dashed curve). As can be seen, the mean trajectories show very small difference until the solid curve approaches its asymptotic position. At this stage the mean trajectory computed from the Lagrangian particle tracking method starts to show random noise in the mean trajectories, indicating a decorrelation of the mean trajectory with the background flow. Figure 5(b) shows the comparison between the standard deviations computed from the different methods in log scale. The curves denote the computation from FP and the markers are from Lagrangian particle tracking. There is no notable difference between the two cases as the mean trajectory reaches its asymptote. For reference we also plot in Fig. 5(b) the function $t^{1/2}$ as the straight line. The plot indicates an asymptotic scaling of the standard deviation similar to Brownian motion.
V. DISCUSSIONS

In this paper we have studied the Lagrangian dynamics of a tracer in an inertia-gravity wave field embedded within an environment of Gaussian white noise. It can be shown that the deterministic Lagrangian trajectories of an IGW are exactly zero over a wave cycle, and hence no Lagrangian mixing occurs in such a case. We examined the question of whether mean drift and nontrivial mixing structure will arise when stochasticity is added to the model. The study of the probability density structure has important implications in understanding the Lagrangian dynamics of a passive tracer in a stochastic environment. In the case considered in this paper, it means that Lagrangian stirring of the idealized IGW does arise in a realistic environment, induced by stochasticity in a nonlinear flow.

We have formulated the problem in the context of an IGW in the upper-troposphere-lower-stratosphere, motivated by studying tracer dynamics leading to better understandings of the water vapor and ozone concentration, which are two important greenhouse gases in these regions. The nondimensional Fokker–Planck equations were solved using a pseudospectral direct numerical simulation (DNS) solver, where the probability density is confined well within the computational box and the periodic boundary conditions mimic a vanishing boundary condition at infinity. We find that, due to the nonlinear background, nontrivial mean trajectories arise when stochasticity is considered, and the difference between the stochastic mean trajectory and the deterministic trajectory increases with the sizes of the variances imposed on the system. These mean trajectories compare well with analytic solutions. Due to this mean motion, we find that there are initial phases where mean trajectories repel or attract nearby mean trajectories, serving as the enhancers in the stirring of the tracers. In the deterministic flow, however, no such structures can be identified. We also studied the long-term behavior of trajectories and find that the trajectories asymptote to positions away from their initial conditions. Long-term behaviors of higher order moments indicate that the process is Gaussian. However, the probability density can have nonzero skewness due to its dependence on initial phase. In addition, analytical solutions of the mean trajectories are obtained and compared with the numerics. The good correlation between our analytical and numerical results gives us confidence in the accuracy of the various statistics estimated from the numerical solutions.

We note that there are several ways to extend the current study. First, we have assumed that the background flow is given by an idealized IGW and thus the interaction with other processes, such as the jet stream which emit these IGW, or wave–wave interactions, should be studied. Stochastic Stokes drift in wave–wave interactions for 1D waves and wave–mean interactions have been discussed in Jansons and Lythe (1998) and Restrepo (2007), where the authors only focused on the mean drift. We will carry further analyses of 3D wave–mean/wave–wave interactions from both numerical and analytical approaches and obtain a more complete picture of the statistics. Such studies of tracer dynamics in nonlinear interactions will reveal better the realistic mixing structure across the tropopause. Second, we have only considered a stable configuration of the IGW. When the background flow becomes unstable, which is usually the case of more concern, Lagrangian mixing is enhanced by the nonlinear motion even in the deterministic flow [Mahalov et al. (2008)]. Random noise will also enter the dynamics of the waves. We want to investigate tracer dynamics in this scenario of more intense mixing to better characterize the statistics. Third, we have assumed that the mean flow is perturbed by Gaussian white noise, which is not the most physically realistic assumption. For eddying motion at the scales considered, the processes are spatially and temporally correlated, hence in the future we will also consider cases with these correlations.

Nevertheless, even with an elementary assumption of Gaussian white noise, our study demonstrates the existence of nontrivial Lagrangian stirring of the tracers. This suggests that the contribution of randomness to Lagrangian dynamics and its applications deserve more attention from scientists interested in the studies of transport processes in nonlinear flows.

ACKNOWLEDGMENTS

We acknowledge support from the Air Force Office of Scientific Research (Grant No. FA-9550-08-1-0055) and the National Science Foundation (Grant No. ATM-0934592). We also thank Bill Young for helpful insights on shear dispersion.

APPENDIX: DERIVATION FOR ANALYTIC SOLUTIONS

The mean trajectory of a particle moving under the influence of the stochastic velocity field can be calculated explicitly [Young et al. (1982)] through the hierarchy of moments. Here we use an approach based on infinitesimal generators. If \( f(X,Y,Z,t) \) is a continuous function of its arguments, then the expectation \( u(t) = \mathbb{E}[f(X_t,Y_t,Z_t,t)] \) solves the integral equation

\[
u(t) = u(0) + \int_0^t \mathbb{E}[A(f(s))]ds,
\]

where

\[
A[f] = f_x + U \cos \Phi f_x + U \sin \Phi f_y - U \cos \Phi f_z + \frac{1}{2}D_h (f_{xx} + f_{yy}) + \frac{1}{2}D_s f_{zz},
\]

is the infinitesimal generator of the diffusion process that solves Eq. (6). Because of the symmetry in \( X \) and \( Z \), we obtain closed systems of equations for the moments of this process due to cancellations of higher order terms in the generator.

To calculate the first order moments, let \( f_x = X, f_y = Y \), and \( f_z = Z \) and observe that

\[
A[f_x] = U \cos \Phi, \quad A[f_y] = U \sin \Phi, \quad A[f_z] = U \cos \Phi.
\]

Thus, taking the expectations, we have
To calculate the expectations appearing inside the integrals, we observe that conditional on the initial values $X_0$ and $Z_0$, and $\Phi$, is normally distributed with mean $2 \pi (X_0 + Z_0 - s)$ and variance $4 \pi^2 D s$ [cf. Eq. (10)]. Thus, if $E_1 = E[\sin \Phi_0]$ and $E_2 = E[\cos \Phi_0]$ denote the expectations with respect to the possibly random initial phase, then a little calculus shows that

$$E[e^{i\phi}] = (E_2 + i E_1) e^{-(2 \pi^2 D + 2 \pi i) \tau^2} = E_{\phi} e^{-(2 \pi^2 D + 2 \pi i) \tau^2}. \quad (A5)$$

$E_{\phi} = E_2 + i E_1 = E[e^{i\phi(0)}]$ is introduced here for use in the derivation of higher order moments. These expressions can then be substituted back into Eq. (A4) to solve for the mean position of the tracer at time $\tau$, given by Eq. (7). Note the similarities and differences between the current approach and method of moments outlined in Young et al. (1982). In their paper, horizontal averages of tracer concentration are obtained and used to solve for higher moments. Here the expectations $E[\sin \Phi]$ and $E[\cos \Phi]$ are elementary quantities in constructing the first moment.

In order to compute the second order moments, we need to evaluate the generator on quadratic functions of $X, Y, Z$. Writing $f_{XX} = X^2, f_{XY} = XY$, etc., we obtain

$$A[f_{XX}] = 2 U X \cos \Phi + D_h,$$

$$A[f_{YY}] = 2 U Y \sin \Phi + D_h,$$

$$A[f_{ZZ}] = -2 U Z \cos \Phi + D_h,$$

$$A[f_{XY}] = U Y \cos \Phi + U X \sin \Phi,$$

$$A[f_{XZ}] = U (Z - X) \cos \Phi,$$

$$A[f_{YZ}] = U Z \sin \Phi - U Y \cos \Phi.$$  \hspace{1cm} (A6)

Thus, we also need to calculate the expectations of $X \cos \Phi, X \sin \Phi, Y \cos \Phi, Y \sin \Phi, Z \cos \Phi$, and $Z \sin \Phi$ to proceed. Writing these terms in the form of complex exponentials, $e_X(s) = X e^{i\Phi s}$, etc., we have

$$A[e_X(s)] = -(2 \pi^2 D + 2 \pi i) X e^{i\Phi s} + \frac{U}{2} (1 + e^{2i\Phi s})$$

$$+ 2 \pi i D e^{i\Phi s},$$

$$A[e_Y(s)] = -(2 \pi^2 D + 2 \pi i) Y e^{i\Phi s} + \frac{U}{2} (1 - e^{2i\Phi s}),$$

$$A[e_Z(s)] = -(2 \pi^2 D + 2 \pi i) Z e^{i\Phi s} - \frac{U}{2} (1 + e^{2i\Phi s})$$

$$+ 2 \pi i D e^{i\Phi s}.$$  \hspace{1cm} (A7)

Taking the expectation of Eq. (A1) and differentiating with respect to time, the solution to Eq. (A7) can be found by solving three systems of linear first order ODEs. These three systems only differ in their forcing terms. Following Eq. (A5), we first obtain the expectations

$$E[e^{2i\Phi}] = E_{2i\Phi} e^{-(8 \pi^2 D + 4 \pi i) \tau},$$

where $E_{2i\Phi} = E[e^{2i\Phi(0)}]$.

Using $g$ to represent the unknowns and $b$ to represent the forcings of the individual systems, we have

$$g = \mu g + b,$$  \hspace{1cm} (A9)

where $\mu = -(2 \pi^2 D + 2 \pi i)$. Individual choices of $(g, b)$ are one of the following three groups: $[E[Xe^{i\Phi}], 2 \pi i D_A E[e^{i\Phi}], U(1 + E[2i\Phi])]$, $[E[Ye^{i\Phi}], U i (1 - E[e^{2i\Phi}]) / 2]$, and $[E[Ze^{i\Phi}], 2 \pi i D_B E[e^{i\Phi}], U(1 + E[2i\Phi]) / 2]$.

The solution to Eq. (A9) is

$$g(\tau) = e^{\mu \tau} \left( E(g(0)) + \int_0^\tau e^{-\mu \tau} b(s) ds \right).$$  \hspace{1cm} (A10)

Using these results we find the analytic expressions for the second order moments shown in Sec. III. For example, using $E_{\chi_2}$ and $E_{\chi_4,\Phi}$ to denote the initial expectations $E[X^2(0)], E[Xe^{i\Phi(0)}]$, we obtain the following expression for the variance:

$$\text{Var}(X) = E[X^2] - \overline{X}^2$$

$$= E_X^2 - \overline{X}^2 + D_h \tau + 2 U \int_0^\tau E[X \cos \Phi] d\tau'$$

$$= E_X^2 - \overline{X}^2 + D_h \tau + 2 U \int_0^\tau \Re[E(Xe^{i\Phi})] d\tau'$$

$$= E_X^2 - \overline{X}^2 + D_h \tau + 2 U \int_0^\tau \Re \left( e^{i\Phi} \int_0^\tau e^{-\mu \tau} b(s) ds \right) d\tau'.$$  \hspace{1cm} (A11)

where $\overline{X}$ is the mean trajectory defined in Eq. (7) and $\Re\{\cdots\}$ denotes the real part. Using the values from Eqs. (A5) and (A8) we obtain the following expression for the variance:

$$\text{Var}(X) = E_X^2 - \overline{X}^2 + D_h \tau + U \Re \left( \frac{2 E_X \Phi}{\mu} (e^{\mu \tau} - 1) - \frac{U \tau}{\mu} \right)$$

$$+ \frac{U}{\mu^2} (e^{\mu \tau} - 1) + \frac{U E_X \Phi}{\mu^2} (e^{\mu \tau} - 1) - \frac{e^{\mu \tau} - 1}{\mu}$$

$$+ 4 \pi i D_B E[\Phi] \left( \frac{e^{\mu \tau} - 1}{\mu^2} + \frac{1}{\mu} \right).$$  \hspace{1cm} (A12)
ments. We therefore stop at the second order and resort to numerical computations to evaluate third- and fourth-order moments.