Problems Part 1

1. Find the sums $1^3 + 2^3 + \cdots + n^3$ and $1^4 + 2^4 + \cdots + n^4$ as functions (polynomials) of $n$.

2. (a) What is the maximal number of pieces into which $n$ straight lines can divide a plane?

(b) What is the maximal number of pieces into which $n$ planes can divide the space?

(c) (Hard problem) What is the maximal number of pieces into which $n$ hyperplanes can divide the $k$-dimensional space? A hyperplane is an affine $(k - 1)$-dimensional subspace, like a 2-dimensional plane in the 3-dimensional space.

3. (a) In a group of people, any two are either friends, or just know each other, or don’t know each other. What should the minimal number of people, to ensure that among them there are at least three with the same relationship (for example three people who are enemies of each other)?

(b) In a group of people, any two are either friends, or enemies, or just know each other, or don’t know each other. What should the minimal number of people, to ensure that among them there are at least three with the same relationship (for example three people who are enemies of each other)?

(c) (Hard problem) Assume that people may have $k$ different relationships. What should the minimal number of people, to ensure that among them there are at least $n$ with the same relationship?

4. Prove that for every natural $n$ and for $x \neq k\pi/2^n$:

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$