(1) Suppose \((1+\sin x)(1+\cos x) = \frac{5}{4}\) and \((1-\sin x)(1-\cos x) = r\), where \(r = \frac{m}{n} - \sqrt{k}\) for some positive integers \(m\), \(n\), and \(k\) with \(\gcd(m, n) = 1\). Find \(m + n + k\).

(2) Find, with proof, the largest possible product of positive integers \(a_1, a_2, \ldots, a_k\) which add up to 2007.

(3) Suppose that \(m\) and \(n\) are integers. For every square \(S\) on the \(m \times n\) board, and every chesspiece \(C\), define \(P_{m,n}(C, S)\) to be the number of squares the chesspiece \(C\) can move to, from square \(S\). Furthermore, define \(P_{m,n}(C)\) (the power of \(C\)) to be the sum of \(P_{m,n}(C, S)\) over all squares \(S\) in the board.

(a) Show that for any square \(S\) and any positive integers \(m\) and \(n\),

\[P_{m,n}(\text{Q}, S) = P_{m,n}(\text{R}, S) + P_{m,n}(\text{B}, S)\]

(b) Are there positive integers \(m\) and \(n\) such that \(P_{m,n}(\text{Q}) = P_{m,n}(\text{R}) + P_{m,n}(\text{B})\)?

(4) Show that if \(n\) can be written as the sum of two squares, then \(5n\) can as well. Try to generalize this result to \(an\) for certain values of \(a\).
(1) In what base $b$ is $221_b$ a factor of $1215_b$?

(2) Let $A$ and $B$ be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

(3) Let $d(n)$ denote the number of positive divisors of $n$, and let $S(n) = \sum_{k|n} d(k)$.

(a) Find all $n$ such that $S(n) = n$, where $n$ is of the form $p^k$, a prime number raised to a nonnegative power.

* (b) Find all $n$ such that $S(n) = n$.

(4) Let $p_k$ be the $k$th prime. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1} - p_n)}$ diverges.
(1) Solve the following equation where $x$ is a positive real number:

$$(8x^3 - 5x^2)(7x^2 - 2x^4) + (9x^3 - 4x^2)(8x^2 - 3x^3)(5x^2 - 2x) = 105x.$$

(2)
(a) Let $a$, $b$, $c$ be positive real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2.$$

Prove

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq 1.$$  
(b) Let $x_0, x_1, \ldots, x_n$ be positive real numbers such that

$$\frac{1}{x_0+1} + \frac{1}{x_1+1} + \cdots + \frac{1}{x_n+1} = n.$$

Prove

$$\frac{1}{n^2x_0+1} + \frac{1}{n^2x_1+1} + \cdots + \frac{1}{n^2x_n+1} \geq 1.$$

(3) There is a machine which will print out a number, after you enter a number that is “acceptable”. The rules that it follows are:

(i) For any number $X$, the number $2X$ is acceptable and produces $X$.
(ii) If $X$ produces $Y$, then $3X$ is acceptable and produces the associate of $Y$.

Here, $MN$ denotes the digits of $M$ followed by the digits of $N$. Hence $2 \cdot 3 = 23$, not 6. The associate of the number $N$ is $N2N$.

(a) Find all numbers $N$ such that, when you enter $N$, you get $N$ as the output.
(b) Find all numbers $N$ such that, when you enter $N$, you get the associate of $N$ as the output.
(c) Show that, for any number $A$, there is a number $N$ such that $N$ produces $AN$.

Given $A$, how do you find this $N$?

(d) Show that there is no number $N$ that produces $N2$ as its output.
(1) A flea lives on the real number line at the number 1. On day 1, the flea will jump to 2. After that, once a day, the flea will jump forward \( k \) units, where \( k \) is a proper divisor of the number it is currently at. For instance, the flea must jump to 3 on the second day, 4 the next, but then can jump to \( 4 + 1 = 5 \) or \( 4 + 2 = 6 \) the third day.

(a) What is the furthest away from home that the flea can get on the \( n \)th day?

(b) Suppose that the flea wants to end up at number \( n \). What is the least number \( v(n) \) of days which are necessary to get there?

(2) Determine all integers \( x, y, z \) such that \( 4^x + 4^y + 4^z \) is a perfect square.

(3) Let \( A, J \) be \( n \times n \) matrices, where \( J \) is the matrix all of whose entries are 1, and let \( c \) be a real number. Let \( C = cJ \), and for \( k = 1, 2, \ldots, n \), let \( A_k \) denote the matrix obtained from \( A \) by replacing each element in row \( k \) with the number \( b \). Prove that

\[
\det(A + C) \det(A - C) = (\det A)^2 - \left( \sum_{k=1}^{n} \det A_k \right)^2.
\]

(4) Let \( S \) be the set of strings made up of the “characters” \( P, N, A, \) and \( - \). A certain machine \( M \) will print out some elements of \( S \), subject to the following conditions:

(a) A string of the form \( P-x \) has the interpretation that “\( M \) will eventually print \( x \).”

(b) A string of the form \( N\!P-x \) has the interpretation that “\( M \) will never print \( x \).”

(c) A string of the form \( P\!A-x \) has the interpretation that “\( M \) will eventually print the associate of \( x \),” where the associate of \( x \) is defined to be \( x-x \).

(d) A string of the form \( N\!P\!A-x \) has the interpretation that “\( M \) will never print the associate of \( x \).”

(e) \( M \) will never print a string whose interpretation is false.

Show that there is a string in \( S \) which is true, but that \( M \) will never print.