(1) Let $A, B, C$ be the angles of a triangle. Prove that

$$\tan^2 \left( \frac{A}{2} \right) + \tan^2 \left( \frac{B}{2} \right) + \tan^2 \left( \frac{C}{2} \right) < 2$$

if and only if $\tan \left( \frac{A}{2} \right) + \tan \left( \frac{B}{2} \right) + \tan \left( \frac{C}{2} \right) < 2$.

(2) A $1 \times 1$ square is tiled by seven tiles similar to a $1 \times \alpha$ rectangle, in the manner shown to the right (one copy is put down with the long side being horizontal, and three are put below it with the long side vertical). Find all possible values of $\alpha$ for which you can do this.

(3) For every natural number $n$, define $S(n)$ to be the unique integer $m$ (if it exists) which satisfies the equation

$$n = \lfloor m \rfloor + \left\lfloor \frac{m}{2!} \right\rfloor + \left\lfloor \frac{m}{3!} \right\rfloor + \cdots$$

(a) Find $S(3444)$, and show that $S(3443)$ does not exist.

(b) Does there exist a number $k$ such that, for any nonnegative integer $n$, at least one of $S(n+1), S(n+2), \ldots, S(n+k)$ exists?

(4) Solve the following equation, where $x$ is a positive real number:

$$(8^x - 5^x)(7^x - 2^x)(6^x - 4^x) + (9^x - 4^x)(8^x - 3^x)(5^x - 2^x) = 105^x$$
(1) Determine, with proof, the largest number which is the product of positive integers whose sum is 2006; i.e., find
\[
\max_{k, a_1, a_2, \ldots, a_k \in \mathbb{Z}^+} a_1 a_2 \cdots a_k.
\]
\[
a_1 + a_2 + \cdots + a_k = 2006
\]

(2) Five points are located on a line. When the ten distances between pairs of points are listed from smallest to largest, the list reads 2, 4, 5, 7, 8, \(k\), 13, 15, 17, 19. Determine the value of \(k\).

(3) Let \(G\) consist of all functions whose domain is the set of all real numbers, which are onto, continuous, and strictly decreasing (if \(g \in G\), then for all real numbers, if \(x < y\) then \(g(x) > g(y))\).

(a) Prove that there is a \(g \in G\) such that \(g(g(x)) = 4x + 5\).
(b) Prove that there is NO \(g \in G\) such that \(g(g(x)) = -4x + 5\).
(c) Prove that if \(g \in G\), then \(g\) has a fixed point; that is, for some real number \(r\) (depending on \(g\)), \(g(r) = r\).
(d) Using (c), prove that there is NO \(g \in G\) such that \(g(g(x)) = x + 1\).
(e) Based on your answers to (a)–(d), for what values of \(a\) and \(b\) is there a \(g \in G\) such that \(g(g(x)) = ax + b\)?

(4) Find all natural numbers \(x\) such that the product of their digits (in base 10 notation) is equal to \(x^2 - 10x - 22\).