(1) Prove that for all positive integers \( n \),
\[
\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(2n-1)(2n)} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.
\]

(2) Suppose \((1+\sin x)(1+\cos x) = \frac{5}{4}\) and \((1-\sin x)(1-\cos x) = r\), where \(r = \frac{m}{n} - \sqrt{k}\) for some positive integers \(m\), \(n\), and \(k\) with \(\gcd(m, n) = 1\). Find \(m + n + k\).

(3) The line segment \(AB\) is tangent at \(A\) to the circle with center \(O\); point \(D\) is inside of the circle; and \(DB\) intersects the circle at \(C\). If \(BC = DC = 3\), \(OD = 2\), and \(AB = 6\), find the radius of the circle.

(4) A dart, thrown at random, hits a square target. Assuming that any two parts of the square with the same area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge.
(1) A right circular cone has a base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

(2) Find \( \lim_{n \to +\infty} \sqrt[n]{n!} \).

(3) Let \( A \) and \( B \) be different \( n \times n \) matrices with real entries. If \( A^3 = B^3 \) and \( A^2B = B^2A \), can \( A^2 + B^2 \) be invertible?

(4) We are given an infinite set of rectangles in the plane, each with vertices of the form \( (0, 0), (0, m), (n, 0) \) and \( (n, m) \), where \( m \) and \( n \) are positive integers. Prove there exist two (distinct) rectangles in the set such that one contains the other.
MAT 194/294/394/494, Fall 2004, Problem Set 2, Group 3

(1) 554 is the base $b$ representation of the square of the number whose base $b$ representation is 24. What is $b$ (written in base 10)?

(2) Let $p$ be a prime greater than 2. Prove that $\frac{2}{p}$ can be expressed in exactly one way in the form $\frac{1}{x} + \frac{1}{y}$ with $x, y$ being positive integers with $x > y$.

(3) Let $S$ be a nonempty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every $a$ in $S$ the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must $S$ be a group?

(4) Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all real numbers $x$. Show that $f(x) = 0$ for all $x$ between $-1$ and 1.