MAT 194/294/394/494, Problem Set 1, Group 1

(1) Graph the relation $\sin x = \sin y$ in the $x, y$-plane.

(2) Find the prime factorization of $1,006,015,020,015,006,001$.

(3) A ball of radius 1 is in a corner touching all three walls. Find the radius of the largest ball that can be fitted into the corner behind the given ball.

(4) A very long hallway has 1000 doors numbered 1 to 1000; all the doors are initially closed. One by one, 1000 people go down the hall: The first person opens every door, the second person closes all doors with even numbers, the third person closes door 3, opens door 6, closes door 9, opens door 12, etc. That is, the $n$th person changes all doors whose numbers are divisible by $n$. After all 1000 people have gone down the hall, which doors are open and which are closed?
(1) The sides of a triangle have lengths 4, 5, and 6. Show that one of its angles is twice another.

(2) Let $S_n$ be the sum of the squares of the first $n$ positive odd integers. What is the units digit of $S_{2004}$? Prove your answer.

(3) A car rode over an ant on the pavement. The ant stuck to the tire for one revolution and then was deposited back onto the pavement. Assuming the radius of the tire is one foot, find the length of the curve travelled by the ant between its death and its final resting place.

(4) Suppose $f$ is a real valued-function which is differentiable. Show that if

$$f(tx) = t \cdot f(x)$$

for all positive $t$, then $f(x)$ is linear. Must $f(x)$ be linear if $f(x)$ is only continuous?
MAT 194/294/394/494, Problem Set 1, Group 3

(1) Find all positive integers which are one more than the sum of the squares of their base ten digits. For example, $35 = 1 + 3^2 + 5^2$.

(2) Let $a_1 = 1$ and $a_{i+1} = \sqrt{a_1 + a_2 + \cdots + a_i}$, for $i > 0$. Determine $\lim_{n \to +\infty} \left( \frac{a_n}{n} \right)$.

(3) Find all integers $A, B, C, D, E$ ($A \leq B \leq C \leq D \leq E$) which, when added in pairs, yield only the sums 401, 546, 691, and 836.

(4) A MAT 210 student in the Tutor Center tried to find the average rate of change of the function $f(x)$ over the interval $[a, b]$ by averaging $f'(a)$ and $f'(b)$. Surprisingly, he got the right answer. Determine all differentiable functions $f(x)$ such that this will always work; i.e., so that

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{2} (f'(a) + f'(b))$$

for all distinct real numbers $a$ and $b$. 