Proof of the Second-Derivative Test

Let \( z = f(x, y) \) be the function graphed in Fig. 2 with \((x_0, y_0)\) as a local maximum point. For fixed values of \( h \) and \( k \), define the function \( g \) by

\[
g(t) = f(x_0 + th, y_0 + tk)
\]

This function records what happens to \( f \) as one moves away from \((x_0, y_0)\) in the direction \((h, k)\), or in the reverse direction \((-h, -k)\).

![Diagram of the function and its second-order conditions](image)

If \( f \) has a local maximum at \((x_0, y_0)\), then \( g(t) \) must certainly have a local maximum at \( t = 0 \). Necessary conditions for this are that \( g'(0) = 0 \) and \( g''(0) \leq 0 \). The first and second order derivatives of \( g(t) \) were calculated in Example 6.1.5. At \( t = 0 \) the second derivative of \( g \) is

\[
g''(0) = f_{xx}(x_0, y_0)h^2 + 2f_{xy}(x_0, y_0)hk + f_{yy}(x_0, y_0)k^2
\]

(1)

So if \( f \) has a local maximum at \((x_0, y_0)\), the expression in (1) must be \( \leq 0 \) for all choices of \((h, k)\).

In this way we have obtained a necessary condition for \((x_0, y_0)\) to be a local maximum point for \( f \). We are more interested in sufficient conditions for local maximum. For the one variable function \( g \) we know that the conditions \( g'(0) = 0 \) and \( g''(0) < 0 \) are sufficient for \( g \) to have a local maximum at \( t = 0 \). It is therefore reasonable to conjecture that we have the following result:

If \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \) and the expression in (1) is \( < 0 \) for all \((h, k) \neq (0, 0)\), then \((x_0, y_0)\) is a (strict) local maximum point for \( f \).

This turns out to be correct: If \( f \) has a strict local maximum in each direction through \((x_0, y_0)\), then this point is a local maximum point for \( f(x, y) \). This will be proved in FMEA. Problem 11 shows that if you delete the word "strict", the conclusion is wrong.