maxima at $x_0$ and $y_0$, respectively. Therefore, $g''(x_0) = f_{xx}(x_0, y_0) \leq 0$ and $h''(y_0) = f_{yy}(x_0, y_0) \leq 0$. On the other hand, if $g''(x_0) < 0$ and $h''(y_0) < 0$, then we know that $g$ and $h$ really do achieve local maxima at $x_0$ and $y_0$, respectively. Stated differently, the conditions $f_{xx}(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$ will ensure that $f(x, y)$ has a local maximum in the directions through $(x_0, y_0)$ that are parallel to the $x$-axis and the $y$-axis.

However, note that the signs of $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ on their own do not reveal much about the behavior of the graph of $z = f(x, y)$ when we move away from $(x_0, y_0)$ in directions other than the two mentioned. Example 6.1.3 illustrates the problem.

It turns out that in order to have a correct second-derivative test for functions $f$ of two variables, the mixed second-order partial $f_{xy}(x_0, y_0)$ must also be considered. The following theorem can be used to determine the nature of the stationary points in most cases. (A proof is given at the end of this section.)

**Theorem 6.2.1 (Second-Derivative Test for Local Extrema)**

Let $f(x, y)$ be a function with continuous second order partials in a domain $S$, and let $(x_0, y_0)$ be an interior point of $S$ that is a stationary point for $f$. Write

$$A = f_{xx}(x_0, y_0), \quad B = f_{xy}(x_0, y_0), \quad \text{and} \quad C = f_{yy}(x_0, y_0)$$

Now:

(i) If $A < 0$ and $AC - B^2 > 0$, then $(x_0, y_0)$ is a (strict) local maximum point.

(ii) If $A > 0$ and $AC - B^2 > 0$, then $(x_0, y_0)$ is a (strict) local minimum point.

(iii) If $AC - B^2 < 0$, then $(x_0, y_0)$ is a saddle point.

(iv) If $AC - B^2 = 0$, then $(x_0, y_0)$ could be a local maximum, a local minimum, or a saddle point.

Note that $AC - B^2 > 0$ in (i) implies that $AC > B^2 \geq 0$, and so $AC > 0$. Thus, if $A < 0$, then also $C < 0$. The condition $C = f_{yy}(x_0, y_0) < 0$ is thus (indirectly) included in the assumptions in (i). The corresponding observation for (ii) is also valid.

The conditions in (i), (ii), and (iii) are usually called (local) second-order conditions. Note that these are sufficient conditions for a stationary point to be, respectively, a strict local maximum point, a strict local minimum point, or a saddle point. None of these conditions is necessary. The result in Problem 6 will confirm (iv), because it shows that a stationary point where $AC - B^2 = 0$ can fall into any of the three categories.

**Example 2**

Find the stationary points and classify them when

$$f(x, y) = x^3 - x^2 - y^2 + 8$$