Sect. 5.4 / Partial Derivatives with Two Variables

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The value of $y$ fixed at $y_0$. The points $(x, y, f(x, y))$ on the graph of $f$ that have $y = y_0$ are those that lie on the curve $K_y$ indicated in the figure. The partial derivative $f_x(x_0, y_0)$ is the derivative of $z = f(x, y_0)$ w.r.t. $x$ at the point $x = x_0$, and is therefore the slope of the tangent line $l_x$ to the curve $K_y$ at $x = x_0$. In the same way, $f_y(x_0, y_0)$ is the slope of the tangent line $l_y$ to the curve $K_x$ at $y = y_0$.

![Figure 9](image)

This geometric interpretation of the two partial derivatives can be explained another way. Imagine that the graph of $f$ describes the surface of a mountain, and suppose that we are standing at point $P$ with coordinates $(x_0, y_0, f(x_0, y_0))$ in three dimensions, where the height is $f(x_0, y_0)$ units above the $xy$-plane. The slope of the terrain at $P$ varies as we look in different directions. In particular, suppose we look in the direction parallel to the positive $x$-axis. Then $f_x(x_0, y_0)$ is a measure of the "steepness" in this direction. In the figure, $f_x(x_0, y_0)$ is negative, because moving from $P$ in the direction given by the positive $x$-axis will take us downwards. In the same way, we see that $f_y(x_0, y_0)$ is a measure of the "steepness" in the direction parallel to the positive $y$-axis. We see that $f_y(x_0, y_0)$ is positive, meaning that the slope is upward in this direction.

Let us now briefly consider the geometric interpretation of the "direct" second-order derivatives $f_{xx}$ and $f_{yy}$. Consider the curve $K_y$ on the graph of $f$ in the figure. It seems that along this curve, $f_{xx}(x, y_0)$ is negative, because $f_x(x, y_0)$ decreases as $x$ increases. In particular, $f_{xx}(x_0, y_0) < 0$. In the same way, we see that moving along $K_x$ makes $f_{yy}(x_0, y)$ decrease as $y$ increases, so $f_{yy}(x_0, y) < 0$ along $K_x$. In particular, $f_{yy}(x_0, y_0) < 0$. (The mixed partials, $f_{xy}$ and $f_{yx}$, do not have such easy geometric interpretations.)

**Example 6** Consider Fig. 10, which shows some level curves of a function $z = f(x, y)$. On the basis of this figure, answer the following questions:

(a) What are the signs of $f_x(x, y)$ and $f_y(x, y)$ at the points $P$ and $Q$? Estimate also the value of $f_x(3, 1)$.

(b) What are the solutions of the equations: (i) $f(3, y) = 4$ and (ii) $f(x, 4) = 6$?

(c) What is the largest value that $f(x, y)$ can attain when $x = 2$, and for which $y$ value does this maximum occur?