MAT 194/294/394/494, Problem Set 2

(1) $ABC$ is a triangle and $D$ is a point on $AB$ produced beyond $B$ such that $BD = AC$, and $E$ is a point on $AC$ produced beyond $C$ such that $CE = AB$. The perpendicular bisector of $BC$ meets $DE$ at $P$. Prove that $\angle BPC = \angle BAC$.

(2) A four-digit number $abcd$ (in base ten) is said to be faulty if the product of the last two digits $c$ and $d$ equals the two-digit number $ab$, while the product of the digits $c - 1$ and $d - 1$ equals the two-digit number $ba$. Determine all faulty numbers.

(3) (a) For any positive integer $n$, prove that there exists a unique $n$-digit number $N$ such that:
   (i) $N$ is formed with only digits 1 and 2; and
   (ii) $N$ is divisible by $2^n$.
   (b) Can the digits “1” and “2” in (a) be replaced by any other digits?

(4) The equation

$$abc = (a + b - c)(a + c - b)(c + b - a)$$

is true if $a = b = c$. Are there any more solutions, where $a$, $b$, and $c$ are positive numbers?
(1) Prove that \( \frac{3^n + (-1)^n}{2} - 2^n \) is divisible by 5 for all \( n \geq 2 \).

(2) Let \( b > 0 \) and \( b^a \geq ba \) for all \( a > 0 \). Prove that \( b = e \).

(3) \( ABC \) is a triangle, and the internal bisectors of \( \angle B, \angle C \), meet \( AC, AB \) at \( D, E \), respectively. Suppose that \( \angle BDE = 30^\circ \). Characterize \( \triangle ABC \).

(4) Find all natural numbers \( x \) such that the product of their digits (in decimal notation) is equal to \( x^2 - 10x - 22 \).
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(1) A student in the Tutor Center tried to find the average rate of change of $f(x)$ over the interval $[a, b]$ by averaging $f'(a)$ and $f'(b)$. Surprisingly, he got the right answer. Determine all differentiable functions $f(x)$ such that this works, i.e., so that

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{2} (f'(a) + f'(b))$$

for all real numbers $a$ and $b$.

(2) Consider the sequence of positive integers: 1, 12, 123, 1234, ..., where the next term is constructed by lengthening the previous term at its right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with “carrying” occurring as in addition: Thus the ninth and tenth terms of the sequence are 123,456,789 and 1,234,567,900, respectively. Determine which terms of the sequence are divisible by 11.

(3) Choose any seven-digit numbers $n_1$, $n_2$, ..., $n_7$ which are multiples of seven, and form a matrix $A$ with entry $(i, j)$ equal to the $10^j$th digit of $n_i$. For instance, if the $n$’s are 8641969, 6135801, 1727607, 2419746, 3197523, 3802470, and 6827177, then

$$A = \begin{pmatrix}
8 & 6 & 4 & 1 & 9 & 6 & 9 \\
6 & 1 & 3 & 5 & 8 & 0 & 1 \\
1 & 7 & 2 & 7 & 6 & 0 & 7 \\
2 & 4 & 1 & 9 & 7 & 4 & 6 \\
3 & 1 & 9 & 7 & 5 & 2 & 3 \\
3 & 8 & 0 & 2 & 4 & 7 & 0 \\
6 & 8 & 2 & 7 & 1 & 7 & 7
\end{pmatrix}.$$  

Prove that $\det A$ is an integral multiple of 7 (no matter what the seven original numbers are).

(4) Evaluate

$$\sqrt{2207 - \frac{1}{2207 - \frac{1}{2207 - \cdots}}}.$$

Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where $a$, $b$, and $c$ are integers.