THE ANTIPODAL LAYERS PROBLEM

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Abstract. For $n > 2k$ and $|n| = \{1, 2, \ldots, n\}$, let the bipartite graph $M_{n,k}$ have vertices
$\{A \subseteq [n] \mid |A| = k \text{ or } n - k\}$ and edges $\{(A, B) \mid A \subseteq B\}$. It has been conjectured that
$M_{2k+1,k}$ (the middle two levels of the Boolean Lattice $Q^{2k+1}$) is Hamiltonian, and we
conjecture the same for arbitrary $n$. Here we show that the conjecture holds for $n$ bigger
than roughly $k^2$, with $k$ large enough. We also define a new product between ranked posets,
giving rise to many new representations of $M_{2k+1,k}$. 
1 Introduction

For odd \( n = 2k + 1 \) let \( \mathcal{M}_k \) denote the middle two levels of the \( n \)-dimensional cube \( Q^n \), i.e., \( \mathcal{M}_k \) is the bipartite graph with vertex set \( \{ S \subseteq [n] \mid |S| = k \text{ or } n - k \} \) and edge set \( \{ (S, T) \mid S \subseteq T \} \). What has come to be known as the \textit{Middle Layers Problem} is the following question. Is \( \mathcal{M}_k \) Hamiltonian? We discuss this problem in section 2, offering many new representations for \( \mathcal{M}_k \), while in section 3 we tackle the following generalization. For \( n > 2k \), \( \mathcal{M}_{n,k} \) is the bipartite graph with vertex set \( \{ A \subseteq [n] \mid |A| = k \text{ or } n - k \} \) and edge set \( \{ (A, B) \mid A \subseteq B \} \). Notice that \( \mathcal{M}_{2k+1,k} \cong \mathcal{M}_k \) (all our isomorphisms will be graph, not poset, isomorphisms). Our conjecture that \( \mathcal{M}_{n,k} \) is Hamiltonian is what we call the \textit{Antipodal Layers Problem}, and we will prove the conjecture is true for \( n > ck^2 + k, k \) large enough.

2 Middle Layers

The \textit{Middle Layers Problem} first appeared in [H] and again as problem 2.5 in [K]. It has been verified [MR] for \( k \leq 11 \). In [KT] the problem is attacked with the understanding that any Hamiltonian cycle in \( \mathcal{M}_k \) is the union of two perfect matchings. There, Kienstead and Trotter generalize the notion of \textit{lexicographic} matchings to \textit{lexical} matchings, in hopes that some pair will work. In [DSW] it is shown that this cannot be the case with lexicographic matchings.

As \( \mathcal{M}_k \) is both vertex-transitive and edge-transitive, one would imagine these properties to aid in the construction of a Hamiltonian cycle. In fact, there are exactly \( \binom{2k+1}{2t+1} \binom{2(k-t)}{k-t} \) isomorphic copies of \( \mathcal{M}_t \) in \( \mathcal{M}_k \) (as vertex-induced subgraphs), and \( \text{Aut}(\mathcal{M}_k) \) acts transitively on each of these families of subgraphs. (That is, given two isomorphic copies, \( \mathcal{M}_t \)
and $M^2$, of $M_t$ in $M_k$, there is some $\sigma \in \text{Aut}(M_k)$ such that $\sigma(M^1) = M^2$.) Although symmetry is exploited to some degree in [A], [CDQ], and [DHR], it seems empirically that the greater the degree of symmetry used by an algorithm, the smaller the cycle it constructs. Babai shows in [Ba] that there is always a cycle of length at least $\sqrt{3m}$ in a vertex-transitive graph on $m$ vertices, and recently Savage [S] has found cycles of length roughly $m^{846}$ in $M_k$.

Induction has been yet another tool to fall short of the task so far. Figures 1 and 2 hint at how the different subgraphs $M_t$, $t < k$, can be embedded in $M_k$. To understand the diagrams, we must define an unusual bowtie product, $\bowtie$, between the Hasse diagrams of two ranked posets. Let the height of a poset be the largest rank (i.e., one less than the length of the longest chain). If $A$ and $B$ are the Hasse diagrams of ranked posets of heights $h + 1$ and $h$, respectively, then we define the product $A \bowtie B$ to have the vertex set

$$V(A \bowtie B) = \left\{(a, b) \mid R_A(a) = R_B(b) \text{ or } R_B(b) + 1 \right\}$$

and edge set

$$E(A \bowtie B) = \left\{ \left\{(a, b_1), (a, b_2) \right\} \mid (b_1, b_2) \in E(B) \right\}$$

$$\cup \left\{ \left\{(a_1, b), (a_2, b) \right\} \mid (a_1, a_2) \in E(A) \right\}.$$ (Here, $R_p(x)$ is the rank of $x$ in the poset represented by $\mathcal{P}$). Notice that $A \bowtie B$ is well defined whereas $B \bowtie A$ is not. Also notice that the bowtie product can be extended to the posets represented by $A$ and $B$ in the obvious way (preserving coordinate-wise order), obtaining a ranked poset with $R_{A \bowtie B}(a, b) = R_A(a) + R_B(b)$.

Given $t \leq n$ and $t \equiv n \mod 2$ let us now define another graph, $\mathcal{N}_t(n)$, to be the Hasse diagram of the poset whose elements are all subsets of $[n]$ of size $s$ for $(n-t)/2 \leq s \leq (n+t)/2$, i.e., the middle $t + 1$ levels of $\mathcal{Q}^n$. As usual, edges are induced by set inclusion. Of course, $\mathcal{N}_1(2k + 1) \cong M_k$ and $\mathcal{N}_n(n) \cong \mathcal{Q}^n$. The following theorem motivates our discussion of these graphs and of our “bowtie product” as well.
**Theorem 1** For $n = 2k + 1$ and $t \leq k$, $\mathcal{M}_k \cong \mathcal{N}_{t+1}(n-t) \bowtie \mathcal{Q}^t$.

The cases of greatest interest are when $t = 2$ or $k$. For $t = 2$ we find two copies of $\mathcal{M}_{k-1}$ at the middle two ranks of our diagram and are smacked with thoughts of induction. In general, for $t = 2s$, we find $\binom{2s}{s}$ copies of $\mathcal{M}_{k-s}$ at the middle two ranks of our diagram. For $t = k$ we see that $\mathcal{M}_k \cong \mathcal{Q}^{k+1} \bowtie \mathcal{Q}^k$ and we hope that Hamiltonian cycles in these cubes might somehow be used to construct one in $\mathcal{M}_k$.

**Proof of Theorem 1.**

Let $S = [n-t]$, $T = \{n-t+1, \ldots, n\}$. We view the vertices of $\mathcal{N} = \mathcal{N}_{t+1}(n-t)$ as the subsets of $S$ of size $s$, with $k-t \leq s \leq k+1$, and the vertices of $\mathcal{Q} = \mathcal{Q}^t$ as the subsets of $T$. Define $f : V(\mathcal{M}_k) \to V(\mathcal{N} \bowtie \mathcal{Q})$ as follows. For $A \subseteq [n]$, $|A| = k$ or $k+1$, let $A_S = A \cap S$, $A_T = A \cap T$, and $A'_T = T \setminus A_T$. Then let $f(A) = (A_S, A'_T)$. To prove that $f$ is a graph isomorphism we first show that it is a vertex isomorphism, then also an edge isomorphism.

In the case of vertices, if $|A| = k$ then $|A_S| = s$ implies that $|A_T| = k-s$ and $|A'_T| = t-k+s$. Also, $R_{\mathcal{N}}(x) = |x| - (k-t)$ since the sets of rank 0 have size $(k-t)$, while $R_{\mathcal{Q}}(x) = |x|$. Thus, $R_{\mathcal{N}}(A_S) = R_{\mathcal{Q}}(A'_T) = t-k+s$, with $k-t \leq s \leq k$. Likewise, if $|A| = k+1$ then $R_{\mathcal{N}}(A_S) - 1 = R_{\mathcal{Q}}(A'_T) = t-k+1+s$, with $k-t+1 \leq s \leq k+1$. Hence $f$ is well-defined, and it is clearly one-to-one and onto.

Now suppose that $(A, B) \in E(\mathcal{M}_k)$ with $|A| = k$ and $|B| = k+1$. Then $B = A \cup \{b\}$ for some $b \in [n] \setminus A$. If $b \in S$ then $B_S = A_S \cup \{b\}$, $B_T = A_T$, and $B'_T = A'_T$, which means that $(B_S, A_S) \in E(\mathcal{N})$ and that $((B_S, A_S), (B'_T, A'_T)) \in E(\mathcal{N} \bowtie \mathcal{Q})$. Similarly, if $b \in T$ then $B_S = A_S$, $B_T = A_T \cup \{b\}$, and $B'_T = A'_T \setminus \{b\}$, so $(B'_T, A'_T) \in E(\mathcal{Q})$ and
$\left( (B_S, A_S), (B_T, A'_T) \right) \in E(\mathcal{N} \bowtie \mathcal{Q})$. Again $f$, being well-defined for edges, is clearly one-to-one and onto. \hfill \square$

As an example, in figure 2, the circled vertices in $\mathcal{N}$, $\mathcal{Q}$, and $\mathcal{M}$ are $B_S = \{1, 2, 5\}$, $B'_T = \{7\}$ (so $B_T = \{6\}$), and $B = \{1, 2, 5, 6\}$, respectively.

3 Antipodal Layers

The Antipodal Layers Problem was first posed by the author during the problem session of the 4th SIAM Conference on Discrete Mathematics, June 1988. It is also noted in [Go] and attributed to Roth as a personal communication. The graph $\mathcal{M}_{n,k}$ is vertex- and edge-transitive but cannot be written so nicely as a bowtie product. Nonetheless, we prove the following theorem.

**Theorem 2** Given any $\varepsilon > 0$ there is an integer $k(\varepsilon)$ such that, for $k > k(\varepsilon)$ and

$$c_1 = \frac{1}{\ln 3} + \varepsilon, \quad c_2 = \frac{2}{\ln 6} + \varepsilon,$$

$\mathcal{M}_{n,k}$ is Hamiltonian for

i) $\binom{n}{k}$ odd and $n \geq c_1 k^2 + k$, and

ii) $\binom{n}{k}$ even and $n \geq c_2 k^2 + k$.

One might notice that a constant of $c = 1/\ln(3/2) = 2.466\ldots$ can be obtained quite easily by using Fact 2, below, on the graph $\mathcal{M}_{n,k}$ itself. Our motivation here was to lower the constant, obtaining $c_1 = 0.9102\ldots$ and $c_2 = 1.116\ldots$. In fact, one might simplify the statement of theorem 2 somewhat to say for example that, for $k \geq 6$, $\mathcal{M}_{n,k}$ is Hamiltonian for odd $\binom{n}{k}$, $n \geq k^2 + k$, and even $\binom{n}{k}$, $n \geq 1.3k^2 + k$. Of course, an improvement in the exponent of $k$ is preferred.
**Proof.**

We will use two well-known facts about graphs. Let $G$ be a graph on $m$ vertices $v_1, \ldots, v_m$ with degrees $d_1, \ldots, d_m$.

**Fact 1.** (Dirac) If $d_i \geq n/2$ for all $i$ then $G$ is Hamiltonian (see [BM]).

**Fact 2.** (Jackson) If $G$ is 2-connected and $d_i = d \geq n/3$ for all $i$ then $G$ is Hamiltonian (see [J]).

The Odd Graph, $O_k$, is defined to be the graph with vertices all $k$-subsets of $[2k+1]$ and edges joining disjoint subsets (see [Bi]). We define the Generalized Odd Graph (also known as Kneser’s Graph), $O_{n,k}$, to have as vertices all $k$-subsets of $[n]$, $n > 2k$, with disjoint subsets adjacent. Notice that $O_{n,k}$ is 2-connected and regular with $d = \binom{n-k}{k}$. In [DHR] it was observed that $M_k \cong O_k \times K_2$, where $x$ is the weak product $((x_1, x_2), (y_1, y_2)) \in E(G_1 \times G_2)$ if and only if $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2))$. Here we observe that $M_{n,k} \cong O_{n,k} \times K_2$ as follows.

For $A, B \subset [n]$, $|A| = |B| = k$, $(A, B) \in E(M_{n,k})$ if and only if $A \cap B = \emptyset$. If we denote $A$ by $(A, 0)$ and $B$ by $(B, 1)$, then clearly we have $((A, 0), (B, 1)) \in E(O_{n,k} \times K_2)$ if and only if $A \cap B = \emptyset$ (i.e., $(A, B) \in E(O_{n,k})$ and $(0, 1) \in E(K_2)$). For convenience we will hereafter let $M = M_{n,k}$ and $O = O_{n,k}$.

First we define a function

$$g : V(M) \to V(O)$$

by

$$g(A) = \begin{cases} A & \text{if } |A| = k \\ \overline{A} & \text{if } |A| = n - k, \end{cases}$$

where $\overline{A} = [n] \setminus A$. Then $g$ is two-to-one since $g(A) = g(\overline{A}) = A$ for $|A| = k$. Likewise, $g$
extended to a function on the edges is two-to-one since, for $|A| = |B| = k$ with $A \cap B = \emptyset$, we have $g((\overline{A}, B)) = g((A, B)) = (A, B)$. When $S$ is a set of edges in $E(O)$ we denote by $g(S)$ its set of preimages in $E(M)$.

Let us take the case that $m = |O| = \binom{n}{k}$ is odd. Then for $n \geq c_1 k^2 + k$ and $k > k(\epsilon)$, we have

$$
\frac{d}{m} = \frac{n-k}{\binom{n}{k}} = \frac{(n-k)k}{(n)_k} > \left(\frac{n-2k}{n-k}\right)^k
$$

$$
= \left(1 - \frac{k}{n-k}\right)^k \geq \left(1 - \frac{1}{c_1 k}\right)^k > \frac{1}{3}.
$$

The last inequality holds because $(1 - 1/c_1 k)^k$ tends to $e^{-1/c_1} = 3^{-1+\epsilon^2}$ as $k$ tends to infinity. Hence, fact 2 tells us we have a Hamiltonian cycle $C$ in $O$. And since $m$ is odd, $g^{-1}(C)$ is a Hamiltonian cycle in $M$. Indeed, if $C = (A_1, A_2, \ldots, A_m)$ then $g^{-1}(C) = (A_1, \overline{A}_2, \ldots, A_m, \overline{A}_1, A_2, \ldots, \overline{A}_m)$.

Figure 3: Combining “half-cycles”.

When $m$ is even $g^{-1}(C)$ splits into two “half-cycles,” $C_1$ and $C_2$, with vertex $A \in C_1$ if and only if $\overline{A} \in C_2$. Figure 3 gives an example, with $n = 8$ and $k = 2$, of how we can combine the two into one Hamiltonian cycle $C_M$ in $M$. We follow $C_1$ from the left to $\{1, 2\}$, jump
across to \( C_2 \) at \( \{5, 6\} \), follow \( C_2 \) left and around to \( \{7, 8\} \), jump back to \( C_1 \) at \( \{3, 4\} \), and continue right and around back to \( \{1, 2\} \).

The crucial ingredient in this construction is the fact that the vertices \( \{1, 2\}, \{3, 4\}, \{5, 6\} \) and \( \{7, 8\} \) are mutually adjacent in \( \mathcal{O} \). The trick will be to guarantee that some Hamiltonian cycle \( \mathcal{C} \) has four such vertices in a row.

Define \( \mathcal{O}' = \mathcal{O}_{n,k} \) as follows. Let \( A = \{1, 2, \ldots, k\} \), \( B = \{k + 1, k + 2, \ldots, 2k\} \), \( C = \{2k + 1, 2k + 2, \ldots, 3k\} \), and \( D = \{3k + 1, 3k + 2, \ldots, 4k\} \) be four mutually adjacent vertices in \( \mathcal{O} \). Let \( V(\mathcal{O}') = \{X\} \cup V' \), where \( V' = V(\mathcal{O}) \setminus \{A, B, C, D\} \). Let \( \mathcal{O}' \) have edges \((E, F)\), with \( E, F \in V' \) and adjacent in \( \mathcal{O} \), and \((E, X)\), with \( E \in V' \) and adjacent to at least two of \( A, B, C \), and \( D \) in \( \mathcal{O} \).

Now if \( \mathcal{O}' \) is Hamiltonian then \( \mathcal{O} \) has a Hamiltonian cycle of the type we need, as the following argument shows. If \( \mathcal{C}' = (\ldots, E, X, F, \ldots) \) is a Hamiltonian cycle in \( \mathcal{O}' \) then, without loss of generality, \((E, A), (F, D) \in E(\mathcal{O}) \). Hence \( \mathcal{C} = (\ldots, E, A, B, C, D, F, \ldots) \) is the desired cycle in \( \mathcal{O} \). By replacing \( \{1, 2\}, \{3, 4\}, \{5, 6\} \) and \( \{7, 8\} \) in Figure 3 by \( A, B, C \), and \( D \) and so on, we see that \( \mathcal{M} \) is Hamiltonian. So it remains to prove that \( \mathcal{O}' \) is Hamiltonian.

If \( E \in V' \) then \( d(E) \geq \binom{n-k}{k} - 3 \), whereas

\[
d(X) = \binom{4}{2} \left( \frac{n-2k}{k} \right) - \binom{4}{1} \left( \frac{n-3k}{k} \right) + \binom{4}{0} \left( \frac{n-4k}{k} \right) \geq 3 \left( \frac{n-2k}{k} \right)
\]

(since \( \binom{n-2k}{k} + \binom{n-4k}{k} \geq 2 \binom{n-3k}{k} \)). So for \( n \geq c_2k^2 + k \) and \( k > k(\varepsilon) \) we have, for all degrees \( d \) in \( \mathcal{O}' \),

\[
\frac{d}{m} > 3 \frac{\binom{n-k}{k}}{\binom{n}{k}} = 3 \frac{(n-2k)_k}{(n)_k} > 3 \left( \frac{n-3k}{n-k} \right)^k
\]

\[
= 3 \left( 1 - \frac{2k}{n-k} \right)^k \geq 3 \left( 1 - \frac{2}{c_2k} \right)^k > \frac{1}{2}
\]
Thus, fact 1 implies that $O'$ is Hamiltonian, which completes the proof. □

4 Remarks

We are constantly reminded by this problem of a conjecture of Lovász (see [L]) which states that every vertex-transitive graph contains a Hamiltonian path. It has also been conjectured that, other than the Peterson and Coxeter graphs and two related to them, all vertex-transitive graphs are Hamiltonian (see [Bo]). Another conjecture of interest is that all Cayley graphs are Hamiltonian (for the definition of a Cayley graph see chapter 16 of [Bi]). Due to the high degree of symmetry in $M_{n,k}(\text{Aut}(M_{n,k}) \cong Z_2 \times S_n$ for all $n > 2k$) we pose the following problem.

**Problem 1.** Determine for which $n$ and $k$ $M_{n,k}$ is a Cayley graph.

In [G] this problem was resolved for $O_{n,k}$. In particular, it was discovered that $O_{2k+1,k}$ is never a Cayley graph (notice that $O_{5,2}$ is the Peterson graph). This was to answer a question first raised by Biggs.

As regards the bowtie product we offer

**Problem 2.** Is it true that for $A$ and $B$ Hamiltonian, $A \bowtie B$ is also Hamiltonian?

In particular, is it true if regularity or vertex-transitivity also holds? Clearly $A$ and $B$ regular or vertex-transitive implies the same of $A \bowtie B$. An affirmative answer would settle the *Middle Layers Problem* but not the *Antipodal Layers Problem*.

Naturally, we must also offer

**Problem 3.** Determine if $N_k(n)$ is Hamiltonian.

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5 References


