Modular elements in the lattice $L(A)$ when $A$ is a real reflection arrangement

H. Barcelo, E. Ihrig*

Department of Mathematics, Arizona State University, Tempe, AZ 85287-1804, USA

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Abstract

Let $W$ be a real reflection group, and let $L_W$ denote the lattice consisting of all possible intersections of reflecting hyperplanes of reflections in $W$. Let $p_W(t)$ be the characteristic polynomial of $L_W$. To every element $X$ of $L_W$ there corresponds a parabolic subgroup of $W$ denoted by $\text{Gal}(X)$. If $W$ is irreducible, we show that an element $X$ of $L_W$ is modular if and only if $p_{\text{Gal}(X)}(t)$ divides $p_W(t)$. This characterization is not true if $W$ is not irreducible. Also, we show that if $W$ is neither $A_n$ nor $B_n$, then the only modular elements are 0, 1 and the atoms of $L_W$.

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1. Introduction

By a (real) reflection group (or Coxeter group) we mean a finite subgroup of $\text{Gl}(n, R)$ which is generated by its reflections. For any given reflection $r$, let $\text{Fix}(r)$ denote the reflecting hyperplane of $r$. For any reflection group $W$, let

$$L_W = \left\{ \bigcap_i \text{Fix}(r_i) \mid \text{where each } r_i \text{ is a reflection of } W \right\}$$

be the lattice of all intersections of the reflecting hyperplanes of $W$ ordered by reverse set inclusion. With this order, $L_W$ is a geometric lattice (see [1], p. 80 for basic properties of geometric lattices) with rank function given by

$$r(X) = n - \dim(X)$$

and with

$$X \vee Y = X \cap Y$$

* Corresponding author. E-mail: ihrig@math.la.asu.edu.

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and

\[ X \land Y = \bigcap \{ Z : Z \in L_W \text{ and } Z \supseteq X \cup Y \}. \]

See [4, p. 23] for more details.

In this paper we assume all lattices are finite geometric lattices. In any such lattice, \( X \) is said to be modular if and only if

\[ r(X) + r(Y) = r(X \lor Y) + r(X \land Y) \]

for all \( Y \) in the lattice. Every lattice has modular elements; every atom is modular, and so are \( \hat{0} \) and \( \hat{1} \) (the minimum and maximum elements of the lattice). We call these modular elements the trivial modular elements of the lattice.

It is the purpose of this paper to characterize the modular elements in the lattice \( L_W \) for every real reflection group \( W \). Theorem 2.1 gives several characterizations of the modular elements of \( L_W \) when \( W \) is an irreducible reflection group. One of these characterizations, Theorem 2.2(c), involves the exponents of a parabolic subgroup \( W \) associated with the modular element. Using this result, we are able to give an exhaustive list of all the modular elements in \( L_W \) for any irreducible \( W \). This is Theorem 2.2(d). In particular, we find that \( L_W \) has only trivial modular elements for any \( W \) which is not of type \( A_n \) or \( B_n \). In Theorem 2.2, we give a result which enables one to find the modular elements in an arbitrary reflection group by finding those in its irreducible components.

The main new tool that we use in the proof of our main theorem is the correspondence between \( L_W \) and the lattice of all parabolic subgroups of \( W \) (see preliminaries of Section 2). It is this correspondence, together with well-known results about parabolic subgroups, which enables us to ultimately find all the modular elements in these lattices.

We conclude this section by briefly describing the context to which these results belong. Associated to every lattice is the characteristic polynomial

\[ p_L(t) = \sum_{X \in L} \mu(\hat{0},X) t^{n-r(X)}, \]

where \( \mu \) denotes the Möbius function of \( L \), and \( n = r(\hat{1}) \) is the rank of the lattice. We shall say that the lattice \( L \) has exponents if \( p_L(t) \) has all rational roots. If \( L \) has exponents, each of the roots of \( p_L(t) \) must be an integer because \( p_L(t) \) is a monic polynomial. In this context, a natural problem which arises is the one of finding some way to determine what the roots of \( p_L(t) \) are.

An important result in this direction is the result of Stanley ([Theorem 4.1 of [6]]) which is central to this paper. A lattice is called supersolvable if it has a chain consisting of \( n + 1 \) modular elements. Such a chain of modular elements is called a maximal modular chain. Stanley's result is that if \( L \) is supersolvable, then \( L \) has exponents. Moreover, if

\[ \hat{0} = X_0 < X_1 < \cdots < X_n = \hat{1} \]
is a maximal modular chain, then the roots of \( p_L(t) \) are given by the integers \( a_i, \)
\( i = 1, \ldots, n \) where \( a_i \) is equal to the number of atoms of \( L \) that are below \( X_i \) but not below \( X_{i-1} \).

An important class of lattices which have exponents are the lattices \( L_W \) defined above for a reflection group \( W \). In this case, the roots of \( p_{L_w}(t) \) are the exponents of the group \( W \). Because of this relationship between the roots of \( p_{L_w}(t) \) and the exponents of the reflection group \( W \), we use the term exponent in a geometric lattice to mean any integer root of \( p_L(t) \). Note that every lattice has one as an exponent. The lattices \( L_W \) themselves belong to a larger class of lattices with exponents; namely, the class of lattices associated to recursively free arrangements (see [4, p. 122]).

The results of this paper relate to the question of finding a direct combinatorial way to discover what the exponents are in a lattice with exponents. In a certain sense this problem remains unresolved even for \( L_W \). There is, as of now, no known simple connection between the exponents of \( W \) and any construct obtainable from the lattice \( L_W \), when \( L_W \) is not supersolvable. However, it has been known for a long time that \( L_W \) is not necessarily supersolvable for all reflection groups \( W \). Recently ([1]), we have determined that the only irreducible \( W \) for which \( L_W \) is supersolvable are the dihedral groups together with the groups of types \( A_n \) and \( B_n \). As far as we know, these are the only groups for which there is a known combinatorial procedure for obtaining the exponents of \( L_w \).

The result of Stanley concerning the factorization of \( p_L(t) \) for supersolvable lattices uses the fact that if any lattice \( L \) has a modular element \( X \) then \( p_{L'}(t) \) divides \( p_L(t) \) where \( L' = [0, X] \). The question which arises is whether any of the factorizations of \( p_{L_w}(t) \) can be explained by the existence of modular elements. Our result shows that when \( W \) is irreducible and \( L_W \) is not supersolvable, modular elements can only be used to explain why \( t-1 \) divides \( p_{L_w}(t) \). Hence, these lattices are the ones for which there is complete factorization of their characteristic polynomial over \( Q \), but for which none of the factorizations of \( p_{L_w}(t)/(t-1) \) is explained by modular elements. In fact, we show that if \( p_{L'}(t) \) divides \( p_{L}(t) \) (where \( L' = [0, X] \), making no assumption about the modularity of \( X \)) then \( X \) must be an atom, 0 or \( \hat{1} \).

2. Modular elements in \( L_W \)

Before we give the main result of the paper (Theorem 2.2), we introduce some notations, and review some of the results we will need.

2.1. Preliminaries

(1) Let \( S \) be a simple system of roots for the reflection group \( W \). Let \( T \subseteq S \). We use \( \langle T \rangle \) to denote the subgroup of \( W \) generated by the reflections corresponding to the roots in \( T \).
(2) A subgroup $H$ of $W$ is called a parabolic subgroup if there is a simple system
of roots $S$ for $W$, and a subset $T$ of $S$ so that $H = \langle T \rangle$.

(3) Let

$$ \mathcal{P}_W = \{ H : H \text{ is a parabolic subgroup of } W \}. $$

Use set inclusion to make $\mathcal{P}_W$ into a partially ordered set (see [2]). (It should be noted
that $\mathcal{P}_W$ is different from the lattice of parabolic subgroups corresponding to one fixed
simple system of roots, $S_0$, which is more commonly mentioned in the literature, and
which has been shown to be isomorphic to the boolean lattice $P(S_0)$. We now recall
the definition of the product of two partially ordered sets which we will use in this
paper (see, for example, [7, p. 100]).

Let $L_1$ and $L_2$ be two (finite geometric) lattices. Give

$$ L_1 \times L_2 $$

(the cartesian product of $L_1$ with $L_2$ ) the partial order defined by

$$ (X_1, Y_1) \leq (X_2, Y_2) \quad \text{iff} \quad X_1 \leq X_2 \text{ and } Y_1 \leq Y_2. $$

$L_1 \times L_2$ is also a (finite geometric) lattice with

$$ (X_1, Y_1) \lor (X_2, Y_2) = (X_1 \lor X_2, Y_1 \lor Y_2), $$

$$ (X_1, Y_1) \land (X_2, Y_2) = (X_1 \land X_2, Y_1 \land Y_2) $$

and with

$$ r(X, Y) = r(X) + r(Y). $$

Let $W$ be a reflection group acting on a vector space $V$. Let $X \in L_W$ and let $H \subseteq W$.

From [2], we recall that

$$ \text{Gal}(X) = \{ g \in W : g(x) = x \quad \text{for all } x \in X \} $$

and

$$ \text{Fix}(X) = \{ v \in V : h(v) = v \quad \text{for all } h \in H \}. $$

The primary tool that we use in this paper, besides the results of Stanley discussed
in the introduction, is the following result which we recently proved in ([2]):

**Theorem 2.1.** \text{Gal}(X) \text{ is a parabolic subgroup for all } X \in L_W, \text{ and the function}

$$ \text{Gal} : L_W \rightarrow \mathcal{P}_W $$

is a lattice isomorphism with inverse $\text{Fix}$. 

This enables us to identify the elements of $L_W$ with parabolic subgroups of $W$, and
we make use of this identification in the proof of our next theorem.
Theorem 2.2. Let $W$ be an irreducible reflection group. Let $L = L_W$, and for $X \in L$ let $L' = [0, X]$. The following are equivalent:

(a) $X$ is a modular element.

(b) $p_L(t)$ divides $p_{L'}(t)$.

(c) The multiplicity of every exponent of Gal($X$) is less than or equal to its multiplicity as an exponent of $W$.

(d) One of the following is true:

(i) $X$ is either $0$, $1$ or an atom.

(ii) $W$ is of type $A_n$ (with roots $e_i - e_j$ for $1 < i, j \leq n + 1$ and $i \neq j$). There are indices $i_0, i_1, \ldots, i_k$ so that

\[ X = \{ (x_1, \ldots, x_{n+1}) : x_{i_0} = x_{i_1} = \cdots = x_{i_k} \} \]

and Gal($X$) is of type $A_{n-k}$.

(iii) $W$ is of type $B_n$ (with roots $e_i \pm e_j$ for $1 \leq i, j \leq n$ and $i \neq j$ and $e_i$ for $1 \leq i \leq n$). There are indices $i_1, \ldots, i_k$ so that

\[ X = \{ (x_1, \ldots, x_n) : 0 = x_{i_1} = \cdots = x_{i_k} \} \]

and, Gal($X$) is of type $B_{n-k}$.

Proof. (a) $\Rightarrow$ (b) follows from Theorem 2 of [5, p. 25].

(b) $\Rightarrow$ (c) We assume (b) is true. Let $a$ be an exponent of Gal($X$) with multiplicity $m$. Hence $(t - a)^m$ divides $p_{L_{[0,X]}}(t)$. We now use Corollary 6.28 of [4, p. 225] which states that $L_{\text{Gal}(X)}$ is isomorphic to the lattice $[0, X]$ . This means $(t - a)^m$ divides $p_{L'}(t)$, which in turn divides $p_L(t)$ by assumption. Thus, $(t - a)^m$ divides $p_L(t)$ which shows that $a$ is an exponent of $L$ with multiplicity at least $m$.

(c) $\Rightarrow$ (d) is the heart of the theorem. We assume that $X$ is an element of $L$ such that the multiplicity of every exponent of Gal($X$) is less than its multiplicity as an exponent of $W$. We will show that $X$ is one of the lattice elements listed in (d).

First, we note that Gal($X$) is an irreducible reflection group. Indeed, for any arrangement 1 is an exponent of multiplicity one if and only if the arrangement is not a direct sum. This means $W$, being irreducible, 1 is an exponent of multiplicity 1. Thus, if Gal($X$) were not irreducible, it would have 1 as an exponent with multiplicity at least 2, contradicting (c).

Next, we begin our task of showing that all $X$ which satisfy (c) are listed in (d). Our initial step is to reduce this to the proof of a statement involving only parabolic subgroups. We let $H = \text{Gal}(X)$. The convenient aspect of condition (c) is that it only involves a constraint on $H$. We say that a parabolic subgroup $H$ has 'compatible exponents' if the multiplicity of every exponent of $H$ is less than or equal to its multiplicity as an exponent of $W$. Thus, with the help of the lattice isomorphisms Fix and Gal, we may reformulate what we must show as follows. We will show that if $H$ is any parabolic subgroup of $W$ with compatible exponents, then Fix($H$) is listed in (d).
We now search for all parabolic subgroups with compatible exponents. We simplify our search in two ways. First, we note that the Coxeter graph of $H$ must be a connected subgraph of a Coxeter graph of $W$. It is a subgraph because $H$ is a parabolic subgroup; it is connected because $H$ is irreducible by the argument given above. The second simplification uses the well known result (see, for example, [3, p. 10]) that if $S$ and $S'$ are two simple systems, there is a $g \in W$ with $gS = S'$. Note that for any set of reflections, $T$, $\langle gT \rangle$ will be a subgroup with compatible exponents if and only if $\langle T \rangle$ is. This is true because $\langle gT \rangle = g\langle T \rangle g^{-1}$, and hence $\langle gT \rangle$ has the same exponents as $\langle T \rangle$.

These results mean that we may focus in on a single simple system of roots in our quest for all parabolic subgroups with compatible exponents. To see how we will do this, let $S$ be one fixed simple system of roots. Assume $H'$ is any parabolic subgroup with compatible exponents. Since $H'$ is parabolic, there is a simple system of roots, $S'$, with subset $T'$ so that $H' = \langle T' \rangle$. As mentioned above, there is a $g \in W$ so that $S' = gS$. Define $T = g^{-1}T' \subseteq S$, and define $H = \langle T \rangle$. $H$ is a parabolic subgroup with compatible exponents, and

$$\text{Fix}(H') = \text{Fix}(\langle gT \rangle) = g \text{Fix}(\langle T \rangle) = g \text{Fix}(H).$$

In other words, in order to find $\text{Fix}(H')$ for every parabolic subgroup $H'$ with compatible exponents, it is enough to find $g \text{Fix}(H)$ where $g \in W$ and $H$ is a parabolic subgroup with compatible exponents of the form $H = \langle T \rangle$ where $T \subseteq S$, a fixed simple system of roots.

More precisely, it suffices to look at the Coxeter graph of $W$ and consider all of its possible connected subgraphs. For each of the connected subgraphs there corresponds a Coxeter group. One then looks at the corresponding sequence of exponents, and checks if it is a compatible sequence or not.

As an example we treat the case of $E_6$. (We use Humphrey's notation [3] for the Coxeter groups.) Fig. 1 gives the Coxeter graph and the sequence of exponents corresponding to $E_6$.

The different possible connected subgraphs together with the corresponding type of Coxeter groups and sequence of exponents are given in Fig. 2.

One sees that in each of these cases, except the first one, no sequence of exponents is compatible with the one of $E_6$. For example, 2 is an exponent of multiplicity 1 for each subgroup $H$ of $E_6$ of type $A_2$, $A_3$, $A_4$, or $A_5$ while of multiplicity 0 for $E_6$. The exponent 3 plays the same role for the $H$'s of type $D_4$ or $D_5$. Thus, the only parabolic subgroup remaining is of type $A_1$. But $\text{Fix}(A_1)$ is a reflecting hyperplane thus an atom of $L_{E_6}$ which is the case covered in (i).
Finally, we discuss (d) ⇒ (a). The elements described in (d) (i) are modular in any lattice. The elements described in (d) (ii) and (d) (iii) are known to be modular (see [5; 4, p. 32]).

There is also a result for a reflection group which is not irreducible which enables one to reduce the problem of finding modular elements to the irreducible case described above.

**Theorem 2.3.** Let \( W \) be a reflection group acting on a vector space \( V \). Let \( W_i \) be irreducible reflection groups acting on vector spaces \( V_i \) for \( i = 1, \ldots, k \). Assume

\[
W = \bigoplus_i W_i
\]

and

\[
V = \bigoplus_i V_i.
\]

Then a parabolic subgroup \( H \) of \( W \) is modular in \( \mathcal{P}_W \) if and only if each of the groups \( W_i \cap H \) are modular in \( \mathcal{P}_{W_i} \).

**Proof.** Under these assumptions, we have that \( \mathcal{P}_W \) has the following decomposition:

\[
\mathcal{P}_W = [\hat{0}, W_1] \times [\hat{0}, W_2] \times \cdots \times [\hat{0}, W_k].
\]

This decomposition is given for \( L_W \) in [4, p. 243], and can be obtained for \( \mathcal{P}_W \) by way of the lattice isomorphism Gal. Observe that modular elements in a product lattice are products of the modular elements of each factor (see [4]). Thus, \( H \) is modular in \( P_W \) if and only if \( W_i \cap H \) is modular in \( P_{W_i} \) for all \( i \). We claim that \( W_i \cap H = W_i \cap H \). Indeed, we have that \( H' \cap H = H' \cap H \) whenever \( H, H' \in P_W \) and \( H' \) is a modular element; \( W_i \) is central thus modular in \( P_{W_i} \). The fact that \( H' \cap H = H' \cap H \) when \( H' \) is a modular element follows by applying the map Gal to the equation \( X' \cap X = X' + X \), which holds [4, p. 30] for elements \( X', X \) of \( L_W \) when \( X' \) is modular in \( L_W \).
As a corollary of this result, we are able to give an example of an element $X$ of $L_W$ for which $p_{[0,X]}(t)$ divides $p_L(t)$ and for which $X$ is not modular.

**Example 2.3.** Let $W$ be any reflection group. Let $H$ be any parabolic subgroup which is not a modular element in $P_W$. Let

$$W' = W \oplus H.$$  

As mentioned above, we may, after an identification, assume

$$P_{W'} = P_W \times P_H.$$  

Let $X \in P_{W'}$ be defined by $X = (H, \hat{0})$. We have

$$p_{L_{W'}}(t) = p_{P_W}(t)p_{L_H}(t),$$

so that $p_{L_{W'}}(t)$ divides $p_{L_H}(t)$. However, Theorem 2.3 says $X$ is not modular in $P_{W'}$ since $H$ is not modular in $P_W$.

**References**