Name:

- You can take up to 2 hours for completing this exam.
- Close book, notes and calculator.
- Do not use your own scratch paper.
- Write each solution in the space provided, not on scratch paper.
- If you need more room, write on the back of the page. If you still need more room, ask for scratch paper.
- Show your reasoning on all problems; do not simply write an answer.
- Your solutions must be complete and organized, otherwise points may be deducted.
- Do all 6 problems. There is a total of 60 points.
1. (a) (5 points) If \( t \in \mathbb{Q} \setminus \{0\} \) prove that \( \phi_t : \mathbb{Q}^+ \to \mathbb{Q}^+ \) given by \( \phi_t(r) = tr \) is an automorphism of the additive group \( \mathbb{Q}^+ \).

**Solution:**

\( \phi_t \) is clearly an additive group homomorphism which, since \( t \neq 0 \), is injective. Since \( \phi_t(t^{-1}x) = x \) we see that \( \phi_t \) is also surjective. Hence \( \phi_t \) is an automorphism.

(b) (5 points) Prove that the only characteristic subgroups of \( \mathbb{Q}^+ \) are \( \{1\} \) and \( \mathbb{Q}^+ \). Recall that a subgroup \( H \) of a group \( G \) is called characteristic if it is carried to itself by all automorphisms of \( G \).

**Solution:** Let \( H \) be a non-trivial characteristic subgroup of \( \mathbb{Q}^+ \). For every \( t \in \mathbb{Q} \setminus \{0\} \) we have \( \phi_t(H) = H \). We show that \( H = \mathbb{Q}^+ \).

Since \( H \) is non-trivial, choose \( x \in H \) with \( x \neq 0 \) and let \( y \in \mathbb{Q} \setminus \{0\} \). Then for \( t = yx^{-1} \neq 0 \) we have

\[
y = tx = \phi_t(x) \in \phi_t(H) = (H),
\]

from which \( H = \mathbb{Q}^+ \) follows.
2. (5 points) Let $G$ be a group of order $p^r m$ where $p$ is a prime integer that does not divide $m$.
Prove that $G$ contains a subgroup of order $p^r$ for every integer $r \leq e$.

Solution:

3. Use Sylow theorems to prove that

(a) (5 points) There is no simple group of order 56.

Solution: See homework solutions.

(b) (5 points) Every group of order 35 is cyclic.

Solution:

Let $G$ be a group of order $35 = 5 \times 7$. The number of Sylow 5-subgroups is congruent to 1 modulo 5 and divides 35. Hence there is only one Sylow 5-subgroup, say $H$, which is therefore normal. A similar argument yields that there is only one Sylow 7-subgroup, say $K$, which is therefore normal. We claim that $G \approx H \times K$. Indeed we have that,

i. $H$ and $K$ are both normal subgroups of $G$.

ii. $H \cap K = \{1\}$ since $|H|$ and $|K|$ are relatively prime integers.

iii. $|HK| = |H||K|/|H \cap K| = 5 \times 7 / 1 = 35$, so $HK = G$.

It follows that $G \approx H \times K \approx C_5 \times C_7 \approx C_{35}$ as required.
4. (10 points) Let $G$ and $H$ be the following two groups:

$$G = \langle a, b \mid a^4, a^2b^{-2}, aba^{-3}b^{-1} \rangle, \quad H = \langle a, b \mid ab^{-1}a^{-1}b^{-1}, ba^{-1}b^{-1}a^{-1} \rangle$$

Prove that $G \approx H$ by showing that the relations of $G$ imply those of $H$ and conversely.

**Solution:**

We show that the relations of $H$ are consequences of those of $G$.

In fact,

$$bab = b^2a^3 \text{ since } ab = ba^3$$
$$= a^2b^3 \text{ since } b^2 = a^2$$
$$= a \quad \text{ since } a^4 = 1.$$

Also

$$aba = ba^3a \text{ since } ab = ba^3$$
$$= b \quad \text{ since } a^4 = 1.$$

Next, we show that the relations of $G$ can be deduced from those of $H$. In fact,

$$a^2 = abab \quad \text{ since } a = bab$$
$$= b^2 \quad \text{ since } aba = b.$$

and

$$ab = bab^2 \quad \text{ since } a = bab$$
$$= ba^3 \quad \text{ since } b^2 = a^2.$$

Finally, $b = aba = ba^3a = ba^4$ so $a^4 = 1$ as required.
5. (a) (5 points) Show that in the symmetric group the product of two transpositions \((ij)(kl)\) can always be written as a product of 3-cycles. Do not forget to treat the case where some of the indices are equal.

**Solution:** In the first case, where \(i, j, k, l\) is a set of 4 distinct integers we see that \((ij)(kl) = (ilk)(ijk)\). In the case where 2 of the integers are equal we have that \((ij)(jl) = (ijl)\). These are the only two possible cases, (besides the identity, in which case any 3-cycle raised to the third power yield the identity) hence every product of 2 transpositions can be written as a product of 3-cycles.

(b) (5 points) Prove that the alternating group \(A_n\) is generated by 3-cycles, if \(n \geq 3\).

**Solution:** Since every permutations in \(A_n\) can be written as a product of an even number of transpositions, we can group the transpositions in pair, and use our result in \(a)\) to write every permutation of \(A_n\) as a product of 3-cycles.
6. Let $G$ be a simple group of order 168.

(a) (5 points) Prove that $G$ has eight Sylow 7-subgroups.

**Solution:** By the third Sylow theorem we can conclude that the number of Sylow 7-subgroups is either 1 or 8. But since $G$ is simple we have that the number of Sylow 7-subgroups is 8.

(b) (5 points) Show that if $P$ is a Sylow 7-subgroup of $G$ then the normalizer $N(P)$ of $P$ has order 21.

**Solution:** By a) above we know that the index of the normalizer $N(P)$ of $P$ in $G$ has index 8. Thus $|N(P)| = 21$ since $168 = 8 \times 21$.

(c) (5 points) Show that $G$ contains no subgroup of order 14.

**Solution:**

Assume that $H$ is a subgroup of $G$ of order 14. Again using the third Sylow subgroup theorem we find that the number of Sylow 7-subgroups of $H$ is 1, and $H$ has a normal Sylow 7-subgroup, say $K$. Since $|K| = 7$, $K$ is also a Sylow 7-subgroup of $G$. Now since $K$ is normal in $H$ we must have that $H$ is a subgroup of the normalizer $N_G(K)$ of $K$ in $G$. This shows that the order of $H$ must divide the order of $N_G(K)$, but this is a contradiction since 14 does not divide 21.