6. Prove that a linear operator on $\mathbb{R}^2$ is a reflection if and only if its eigenvalues are 1 and $-1$, and its eigenvectors are orthogonal.

Answer:
If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection along the line $\ell$, then let $v$ be a vector along that line and let $u$ be a vector in $\mathbb{R}^2$ orthogonal to $\ell$. Then $\{u, v\}$ form a basis for $\mathbb{R}^2$ and $T(v) = v$ while $T(u) = -u$, thus the eigenvalues for $T$ are 1, $-1$ and the eigenvectors are $cu$, and $dv$, for any $c, d \in \mathbb{R}$ which are orthogonal by construction. Conversely, if $T$ is a linear operator on $\mathbb{R}^2$ with eigenvalues 1 and $-1$ associated to $v$ and $u$ (respectively), then $T(cv) = cT(v) = cv$ for all $c \in \mathbb{R}$. Hence the line $\ell$ passing through $(0, 0)$ and $v$ is left fixed by $T$, while every vector $w$ orthogonal to $\ell$ is reflected across $\ell$ since $w = du$ and $T(du) = -du$. Thus $T$ is a reflection.

Question: 11. a) Compute the eigenvalues and eigenvectors of the linear operator $m = pr^\theta$.

Answer:
a) Geometrically a reflection followed by a rotation will not change the length of any vectors in $\mathbb{R}^2$. Thus, the only possible eigenvalues are 1, $-1$. If $\theta = 0$, then we saw that 1, $-1$ are the 2 eigenvalues. Moreover, since $r$ is the reflection through the $x$-axis, the eigenvectors associated to 1 are $(a, 0)$ for $a \in \mathbb{R}$, while the one associated with $-1$ are $(0, b)$ for $b \in \mathbb{R}$. If $\theta \neq 0$, then geometrically (see Figure (a)) one sees that if $\ell$ is the line that bisect $\theta$ then for any $p$ on the line $\ell$ we have that $pr^\theta(p) = p$. So any vector on the line bisecting the angle $\theta$ is an eigenvector with eigenvalue 1.

Similarly, one sees (Figure (b)) that any vector on the line $\ell'$ bisecting the angle $\pi - \theta$ will be an eigenvector with eigenvalue $-1$.

Question: c) Do the same thing as in (b) geometrically.

Answer:
c) From Figure (a) for $m$ to be a reflection it must send all the vector of the line $\ell$ to themselves which we already have by $a$. Moreover, for all vectors $v$ on the line perpendicular to $\ell$ we must have $m(v) = -v$. But, by Figure (b) we see that for all vectors $w$ on $\ell'$ $m(w) = -w$. So it remains to show that the line bisecting the angle $\pi - \theta$ is perpendicular to $\ell$. By construction the angle between $\ell$ and $\ell'$ is $\frac{\pi - \theta}{2} + \frac{\theta}{2}$ which is $\pi / 2$. 