1. Prove that for every positive integer \( n \),
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}
\]

**Solution:** Use induction. Let \( P(n) \) be the proposition that \( 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \). Then \( P(1) : 1 \cdot 2 \cdot 3 = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} \) is true. Assume \( P(k) \) is true for \( k \geq 1 \). Show \( P(k+1) \) is true.

**We want to show**
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3)
\]
\[
= \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \text{this is } P(k+1)
\]
\[
= \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \text{by the IH}
\]
\[
= \frac{(k+1)(k+2)(k+3)(k+4)}{4} \quad \text{true.}
\]

2. Prove that 3 divides \( n^3 + 2n \) whenever \( n \) is a positive integer.

**Solution:** Use induction. Let \( P(n) \) be the proposition that \( 3|n^3 + 2n \).
Then \( P(1) : 3|1^3 + 2 \cdot 1 \) is true. Assume \( P(k) \) is true for \( k \geq 1 \), that is, \( k^3 + 2k = 3m \) for some integer \( m \). Show \( P(k+1) \) is true. Show \( 3((k+1)^3 + 2(k+1)) \), that is, show \( 3(k^3 + 2k^2 + 3k + 3) = 3m + 3(k^2 + 3k + 3) = 3(m + k^2 + k + 1) \), so we are done.

3. Evaluate
\[
\sum_{k=0}^{200} \left( \frac{1}{3} \right)^k
\]
Leave your answer as a fraction, do not simplify and do not evaluate exponents.

**Solution:**
\[
\frac{(1/3)^{200} - 1}{(1/3) - 1}
\]
1
4. Is the set \( S = \{ x | x \text{ is an odd integer} \} \) finite? Countable? Uncountable? Prove your claims. **Solution:** It is countable, clearly not finite. We need an injective map from \( S \) to \( \mathbb{Z}^+ \).

\[
f(x) = \begin{cases} 
  x - 1 & \text{if } x > 0 \\
  -x & \text{if } x \leq 0
\end{cases}
\]

5. Let \( p \) and \( q \) be distinct primes, \( a \) and \( b \) arbitrary integers. Show that if \( a \equiv b \pmod p \) and \( a \equiv b \pmod q \), then \( a \equiv b \pmod {pq} \). **Solution:**

Since \( a \equiv b \pmod p \), we know \( p \mid (a - b) \). Since \( a \equiv b \pmod q \), we know \( q \mid (a - b) \). Let \( p_i, a^2, \ldots, p_k \) be the prime factorization of \( a - b \). Then \( p^i \equiv q^i \pmod i \), for some \( i \), \( i \not\equiv j \) since \( p \not\equiv q \). We know this \( i \) and \( j \) must exist because all primes which divide \( a - b \) appear in its prime factorization. Thus \( a - b = p^i \cdot q^i \cdot p_j \cdot \ldots \cdot p_k \Rightarrow p^i \mid (a - b) \) so that \( a \equiv b \pmod {pq} \).

6. Describe an algorithm which finds the second largest element among the distinct positive integers \( a_1, a_2, \ldots, a_n \). Give a good big-O estimate on how many comparisons does your algorithm make in the worst case? (I don't care if your algorithm is not the most efficient one.)

**Solution:** First find the largest element with the algorithm we learned. Remove the largest element from the set. Find the largest element in the smaller list. This will be the second largest element.

The first part takes \( n - 1 \) comparisons, the second part \( n - 2 \) comparisons, so together we need \( 2n - 3 \) comparisons, which is \( O(n) \).

**procedure** `second-max(a_1, \ldots, a_n)`

\[
max_{\text{ind}} := 1 \\
\text{for } i := 2 \text{ to } n \text{ do } \\
\text{if } a_{\text{max ind}} < a_i \text{ then } max_{\text{ind}} := i \\
sec_{\text{max}} := 1 \\
\text{for } i := 2 \text{ to } max_{\text{ind}} - 1
\]
if \( a_{\text{sec\_max}} < a_i \) then \( \text{sec\_max} := i \)
for \( i := \text{max\_ind} + 1 \) to \( n \)
if \( a_{\text{sec\_max}} < a_i \) then \( \text{sec\_max} := i \)
\{ \( a_{\text{sec\_max}} \) is the second largest element \}

(There is a faster method which uses only \( n - 1 + \log_2 n \) comparisons:

*Like a tennis elimination tournament:*

Make pairs of the numbers, in each pair the larger number is the winner. If one of the element has no pair, it is a winner. Make pair of the winners, etc. The final champion is the largest number. The second largest number must be among those who lost to the champion. There are \( O(\log_2 n) \) of them. Find the largest element among those by the usual method.)

7. Use the Euclidean algorithm to find \( \gcd(679, 553) \). \( \gcd = \text{greatest common divisor} \)

**Solution:**

\[
\begin{align*}
679 & = 1 \cdot 553 + 126 \\
553 & = 4 \cdot 126 + 49 \\
126 & = 2 \cdot 49 + 28 \\
49 & = 1 \cdot 28 + 21 \\
28 & = 1 \cdot 21 + 7 \\
21 & = 3 \cdot 7
\end{align*}
\]

The last non-zero remainder is 7, so \( \gcd(679, 553) = 7 \).

8. If the product of two integers is \( 2^7 \cdot 3^4 \cdot 5^2 \cdot 7^{11} \) and their greatest common divisor is \( 2^3 \cdot 3^1 \cdot 5 \), what is their least common multiple?

**Solution:** If \( a = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \) and \( b = 2^{b_1} \cdot 3^{b_2} \cdot 5^{b_3} \cdot 7^{b_4} \) then we know that \( \alpha_1 + \beta_1 = 7 \) and \( \min(\alpha_1, \beta_1) = 3 \), therefore \( \max(\alpha_1, \beta_1) = 4 \).

Similarly we can derive that \( \max(\alpha_2, \beta_2) = 8 - 4 = 4 \), \( \max(\alpha_3, \beta_3) = 2 - 1 = 1 \) and \( \max(\alpha_4, \beta_4) = 11 - 0 = 11 \). Therefore the least common multiple is \( 2^4 \cdot 3^4 \cdot 5^1 \cdot 7^{11} \).
9. List all positive integers which are relatively prime to 36. **Solution:**

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 \\
\end{array}
\]

36 = \(3^2 \cdot 2^2\) so cross out all multiples of 2 and 3. Rest all relatively prime to 36.

10. Use the modular exponentiation algorithm to find \(3^{200} \mod 7\). **Solution:**

\[
3^{200} \equiv 200 = 128 + 64 + 8 = 2^7 + 2^6 + 2^3 \\
3^{200} \mod 7 = 2 \cdot 4 \cdot 2 \equiv 3 \mod 7
\]

11. What is the binary expansion of 100? **Solution:**

\[
100 = 64 + 36 \\
= 64 + 32 + 4 \\
= 2^6 + 2^5 + 2^1 = 1100100
\]
13. Find the least integer \( n \) such that
\[
\frac{x^4 + x^2 + 1}{x^3 + 1}
\]
is \( O(x^n) \)

\( n = 1 \). Show that
\[
\frac{x^4 + x^2 + 1}{x^3 + 1} \quad \text{is} \quad O(x)
\]

\( x^3 + 1 \geq x^2 \) \( \Rightarrow \) \( \frac{1}{x^2 + 1} \leq \frac{1}{x^3} \)

\( x^4 + x^2 + 1 \leq x^4 + x^4 + x^4 = 3x^4 \) \( (x \geq 1) \)

\( \Rightarrow \) \( \frac{x^4 + x^2 + 1}{x^3 + 1} \leq 3x^4 \cdot \frac{1}{x^3} = 3x \)

\( C = 3, \quad k = 1 \)

Show next that \( \frac{x^4 + x^2 + 1}{x^3 + 1} \) is not \( O(\cdot) \)

Suppose there is an \( C \) \( \& \) a \( k \) s.t.
\[
\frac{x^4 + x^2 + 1}{x^3 + 1} \leq C \quad \text{when} \quad x > k. \quad \text{Assume}
\]

Note that when \( x \geq 1 \) \( x^3 + 1 \leq 2x^3 \) so that
\[
\frac{1}{x^3 + 1} \leq \frac{1}{2x^3}
\]
and that \( x^4 + x^2 + 1 \geq x^4 \). Thus,

\( \frac{x^4 + x^2 + 1}{x^3 + 1} \geq \frac{x^4}{2x^3} = \frac{x}{2} \)

To contradict the existence of such a \( C \) \( \& \) \( k \), pick
\( x > \max(1, k, 2C) \). Then
\[
\frac{x^4 + x^2 + 1}{x^3 + 1} \geq \frac{x}{2} > \frac{2C}{2} = C.
\]