Stable Periodic Orbits for a Class of Three Dimensional Competitive Systems

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It is shown that for a dissipative, three dimensional, competitive, and irreducible system of ordinary differential equations having a unique equilibrium point, at which point the Jacobian matrix has negative determinant, either the equilibrium point is stable or there exists an orbitally stable periodic orbit. If in addition, the system is analytic then there exists an orbitally asymptotically stable periodic orbit when the equilibrium is unstable. The additional assumption of analyticity can be replaced by the assumption that the equilibrium point and every periodic orbit are hyperbolic. In this case, the Morse-Smale conditions hold and the flow is structurally stable. © 1994 Academic Press, Inc.

INTRODUCTION

In a series of recent papers, [3–6], M. W. Hirsch showed that an $n$-dimensional competitive system of autonomous ordinary differential equations behaves like a general system of dimension $n-1$ in the sense that every solution tends to a solution of the same system, restricted to a family of invariant $n-1$ dimensional Lipschitz manifolds. In the case $n = 3$, Hirsch [6] shows that the Poincaré–Bendixson theorem holds: a compact limit set (positive or negative) of a three dimensional competitive system containing no equilibria is a periodic orbit. See [13] for additional analogies between three dimensional competitive systems and planar systems.

In the present paper we show that the following result for planar autonomous systems essentially extends to three dimensional competitive systems.

THEOREM A. Let a continuously differentiable, dissipative, planar system of ordinary differential equations have a unique equilibrium point. If that point is nondegenerate then either it is stable or there exists an orbitally

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stable periodic orbit. If, in addition, the planar system is real analytic then there exists an orbitally asymptotically stable periodic orbit when the equilibrium is unstable.

By a nondegenerate equilibrium we mean that the Jacobian matrix at that point is nonsingular. Precise definitions of dissipative and competitive systems are given below. We have been unable to find a reference for Theorem A, which also holds when the plane is replaced by a simply connected open subset of the plane. Theorem A is not difficult to prove (see [15] for a proof) but requires more work than one might initially expect (Zorn’s lemma is used). The second assertion follows from an application of Poincaré’s lemma [11].

Orbitally asymptotically stable periodic orbits are observable in nature and in numerical simulations and therefore easily verifiable sufficient conditions for their existence are important. Theorem A and its analog for three dimensional competitive systems, and Theorems 1.1 and 1.2 below, provide such sufficient conditions.

We consider the system of differential equations

\[ x' = f(x), \quad x \in D, \tag{1.1} \]

where \( D \) is an open subset of \( \mathbb{R}^3 \) and \( f \) is twice continuously differentiable in \( D \). The noncontinuable solution of (1.1) satisfying \( x(0) = x_0 \) is denoted by \( x(t, x_0) \), the positive (negative) semi-orbit through \( x_0 \) is denoted by \( \phi^+(x_0) \) (\( \phi^-(x_0) \)), and the orbit through \( x_0 \) is denoted by \( \phi(x_0) = \phi^-(x_0) \cup \phi^+(x_0) \). We use the notation \( \omega(x_0) = (z(x_0)) \) for the positive (negative) limit set of \( \phi^+(x_0) \) (\( \phi^-(x_0) \)) provided the latter semi-orbit has compact closure in \( D \).

System (1.1) is said to be cooperative in \( D \) if the Jacobian matrix of \( f \) at \( x \), \( Df(x) \), has nonnegative off-diagonal elements

\[ \frac{\partial f_i}{\partial x_j}(x) \geq 0, \quad i \neq j \]

at each point of \( D \). It is said to be competitive in \( D \) if the reverse inequalities hold. System (1.1) is said to be cooperative (competitive) and irreducible in \( D \) provided the Jacobian matrix is an irreducible matrix at each point \( x \in D \) and (1.1) is cooperative (competitive) in \( D \). Recall that an \( n \times n \) matrix \( A \) is irreducible if for each nonempty proper subset \( I \) of \( N = \{1, 2, ..., n\} \) there exist \( i \in I \) and \( j \in N – I \) such that \( A_{ij} \neq 0 \).

For vectors \( x \) and \( y \) in \( \mathbb{R}^3 \) the inequality \( x \preceq y \) (\( x \leq y \)) means that \( x_i \leq y_i \) holds for all \( i \) and \( x \preccurlyeq y \) means that \( x \leq y \) but \( x \neq y \). Two vectors \( x \) and \( y \) are related if either \( x \preceq y \) or \( y \preceq x \) and unrelated otherwise. A set \( A \) is said to be unordered if it does not contain two distinct related points.
The open set $D$ is said to be $p$-convex [4] provided that for every $x$ and $y$ belonging to $D$ for which $x \leqslant y$ the line segment joining $x$ and $y$ belongs to $D$. If $x \leqslant y$, let $[x, y] \equiv \{z : x \leqslant z \leqslant y\}$.

The distinguishing feature of a cooperative (competitive system) in a $p$-convex domain is that the map $x(t, \cdot)$ preserves the partial order relation $\leqslant$ for $t > 0$ ($t < 0$) provided it is defined [4]. If (1.1) is cooperative (competitive) and irreducible and $x < y$ then $x(t, x) \leqslant x(t, y)$ for $t > 0$ ($t < 0$) [4, 12].

In order to state our main results, we introduce the following hypotheses.

(H1) System (1.1) is dissipative: For each $x \in D$, $\phi^+(x)$ has compact closure in $D$. Moreover, there exists a compact subset $B$ of $D$ with the property that for each $x \in D$ there exists $T(x) > 0$ such that $x(t, x) \in B$ for $t \geqslant T(x)$.

(H2) System (1.1) is competitive and irreducible in $D$.

(H3) $D$ is an open, $p$-convex subset of $\mathbb{R}^3$.

(H4) $D$ contains a unique equilibrium point $x^*$ and $\det(Df(x^*)) < 0$.

The main results of this paper follow.

**Theorem 1.1.** Let (H1) through (H4) hold. Then either

(a) $x^*$ is stable, or

(b) there exists a nontrivial orbitally stable periodic orbit in $D$.

**Theorem 1.2.** In addition to the assumptions of Theorem 1.1, assume that $f$ is analytic in $D$. If $x^*$ is unstable then there is at least one but no more than finitely many periodic orbits for (1.1) and at least one of these is orbitally asymptotically stable.

Some remarks will clarify our hypotheses and the statement of Theorems 1.1 and 1.2. It is well known that the dissipation hypothesis (H1) implies the existence of a maximal compact invariant set for the flow of (1.1) which uniformly attracts compact subsets of $D$ [2]. Hereafter we denote this attractor by $A$. The hypothesis (H4) implies that if $x^*$ is unstable then either it is hyperbolic (no eigenvalues of $Df(x^*)$ have zero real part) and $x^*$ has a two dimensional unstable manifold or $x^*$ is nonhyperbolic due to a nontrivial pair of purely imaginary eigenvalues (plus a negative real eigenvalue).

We let $\lambda_i$, $i = 1, 2, 3$, denote the eigenvalues of $Df(x^*)$, ordered such that $\Re \lambda_1 \leqslant \Re \lambda_2 \leqslant \Re \lambda_3$. Observe that $-Df(x^*) + \mu I$ is a nonnegative, irreducible matrix for all large $\mu$ so by the Perron–Frobenius theorem, $-Df(x^*)$ has a real eigenvalue strictly larger than the real part of any other eigenvalue. It follows that $\lambda_1 = \Re \lambda_1 < \Re \lambda_2$. 

It is worth mentioning that if $D$ is homeomorphic to $\mathbb{R}^3$ and (H1) holds then the Brouwer degree of $f$ with respect to any bounded open subset of $D$ containing all equilibria must, by a slight modification of [7, Lemma 2; note the nonstandard index used on page 1020], be minus one (more generally, $(-1)^n$ in $\mathbb{R}^n$). If, in addition, $x^*$ is the unique equilibrium of (1.1) in $D$ and if it is nondegenerate, then by the definition of the Brouwer degree, the inequality $\det Df(x^*) < 0$ must hold. Thus, the reverse inequality in (H4), i.e., $\det Df(x^*) > 0$, would be inconsistent with (H1) unless $D$ is not homeomorphic to $\mathbb{R}^3$; that is, unless the geometry is untypically complicated. Note that in Theorem A, the above arguments imply that $\det Df(x^*) = +1$.

The stronger conclusions of Theorem 1.2 can be obtained by replacing the analyticity assumption by the assumption that $x^*$ and every periodic orbit are hyperbolic (i.e., the real parts of the Floquet exponents do not vanish). A stronger result is stated below. In practice, of course, this hyperbolicity assumption cannot be verified.

**Theorem 1.3.** In addition to (H1)–(H4), assume that $x^*$ and every periodic orbit of (1.1) are hyperbolic. Then the number of periodic orbits is finite and either $x^*$ is asymptotically stable or there exists an orbitally asymptotically stable periodic orbit. In addition, the Morse–Smale conditions hold:

(i) The stable and unstable manifolds of critical elements (equilibria and periodic orbits) intersect transversally.

(ii) Every compact limit set in $D$ is an equilibrium or a periodic orbit.

Two consequences of Theorem 1.3 are important to mention. First, it follows from (ii) of the Morse–Smale conditions that $A$ is the (finite) union of critical elements and their unstable manifolds. Second, by an argument of Hirsch [6, Thm. 3, and discussion following the proof of Thm. 4], there is a neighborhood $U$ of $A$ in $D$ having a smooth boundary such that $f$ is transverse to the boundary of $U$ and the flow of (1.1) restricted to the closure of $U$ is structurally stable.

Our proofs of Theorem 1.1, 1.2, and 1.3 rely heavily on results of M. W. Hirsch in [3–6] and Smith [12, 13].

A number of three dimensional systems arising in applications can be transformed to competitive systems by a simple change of variable [12]. Examples include a model of two competing microbial species in a chemostat with inhibitor [17], the Field–Noyes model for the Belousov–Zhabotinski chemical reaction [19], and a model of cellular control of protein synthesis [18]. Applications of our results to these systems are discussed in [15]. See also [14]. See also [8, 10].
PROOF OF MAIN RESULTS

For the remainder of this paper we assume that (H1) through (H4) hold. By virtue of (H1), for each \( x \in D \), \( x(t, x) \) is defined on an interval \((a(x), \infty)\) where \(-\infty < a(x) < 0\). Recall that \( A \) denotes the maximal compact invariant set implied by (H1).

We note here several further implications of (H1) which are used below. First, \( a(x) = -\infty \) for each \( x \in A \); if \( x \in D - A \) then for each compact subset \( L \) of \( D \), there exists \( S = S(x, L) \) with \( S > a \) such that \( x(t, x) \notin L \) for \( a < t < S \). Hence for each \( x \in D, \phi^-(x) \) has compact closure in \( D \) if and only if \( x \in A \) and in this case, the closure of \( \phi^-(x) \) is contained in \( A \). Obviously, \( x^* \in A \) and any periodic orbits must belong to \( A \). If \( \gamma \) denotes \( x^* \) or a periodic orbit of (1.1) then the unstable manifold of \( \gamma \), \( W^u(\gamma) \), must also belong to \( A \).

**Proposition 2.1.** \( A \) is unordered.

**Proof.** Suppose for contradiction that there exist \( a \) and \( b \) in \( A \) such that \( a < b \). If \( a < x < b \), then by the monotonicity of the flow, \( x(t, a) \ll x(t, x) \ll x(t, b) \) for all \( t < 0 \). Since \( a \) and \( b \) belong to \( A \), the negative orbits through \( a \) and \( b \) have compact closure in \( A \), hence so does the negative orbit of \( x \). By [4, Lemma 4.3], all but an at most countable subset of points on the line segment joining \( a \) to \( b \) must be quasi-convergent points for the time reversed flow. Obviously, these quasi-convergent points must converge to \( x^* \) as \( t \) tend to \(-\infty \). If \( x \) and \( y \) are two points on this segment with \( x < y \), then \( x(t, x) \ll x(t, y) \) for \( t < 0 \) and by the monotonicity of the time reversed flow, \( x(z) = x^* \) for every point \( z \) in the order interval \([x(t, x), x(t, y)]\). As \( D \) is open and \( p \)-convex, we can find \( x \) and \( y \) on the line segment joining \( a \) to \( b \) and \( t < 0 \) sufficiently near zero such that the order interval is contained in \( D \). This order interval is uniformly attracted to \( x^* \) for the time reversed flow. But this would imply that \( \lambda_i \geq 0 \) (see [4, Sect. 2]) and contradict (H4).

**Proposition 2.2.** Let \( w \) be a positive unit vector and \( \Pi \) be the orthogonal projection onto the plane that is orthogonal to \( w \). Then \( \Pi|_A \), the restriction of \( \Pi \) to \( A \), is injective and \( (\Pi|_A)^{-1} \) is Lipschitzian. If the hyperplane is identified with \( \mathbb{R}^2 \), there exists a Lipschitz vector field, \( Y \), on \( \mathbb{R}^2 \) such that \( \Pi \) maps orbits in \( A \) onto orbits of \( Y \) in \( \mathbb{R}^2 \), respecting parameterization. Thus \( \Pi(A) \) is a compact invariant set for the flow generated by \( Y \).

**Proof.** Hirsch proves this result for a limit set (see [3, Theorem A]), but the same argument applies to any unordered, compact, invariant set.
For the remainder of this paper, we fix a positive vector \( w \) and denote by \( \Pi \) the corresponding projection as in Proposition 2.2. We identify the plane orthogonal to \( w \) with \( \mathbb{R}^2 \) and we use the notation for orbits and limit sets, developed for (1.1), for the corresponding objects associated with the vector field \( Y \) on \( \mathbb{R}^2 \).

Theorem 1.1 is proved by obtaining a contradiction to the assumption that \( x^* \) and all periodic orbits of (1.1) are unstable. Therefore, until further notice, we assume:

\[(H5) \quad x^* \text{ is unstable and any periodic orbit of (1.1) in } D \text{ is orbitally unstable.} \]

In view of (H4) and (H5) there are two possibilities for the eigenvalues \( \lambda_i \) of \( Df(x^*) \). Either they are all real and \( \lambda_1 < \lambda_2 \leq \lambda_3 \) or \( \lambda_1 < 0 \) and \( \lambda_2, \lambda_3 = \alpha \pm i\beta, \beta > 0 \) and \( \alpha \geq 0 \). Hereafter, we consider only the second case as the proof in the first case is similar.

**Proposition 2.3.** Exactly one of the following holds:

(i) \( x^* \) is a nonhyperbolic saddle; that is, \( \lambda_1 < 0 = \alpha \) and \( x^* \) is an unstable spiral point on any two dimensional center manifold, \( W^c(x^*) \), of \( x^* \). \( W^u(x^*) \) is contained in \( A \).

(ii) \( x^* \) is a hyperbolic saddle; that is, \( \lambda_1 < 0 < \alpha \) and \( x^* \) is an unstable spiral point on its two dimensional unstable manifold, \( W^u(x^*) \), which is contained in \( A \).

**Proof.** In the hyperbolic case, \( \alpha > 0 \), the unstable manifold of \( x^* \) is contained in \( A \) since it consists of orbits which have compact closure in \( D \) and \( x^* \) is obviously an unstable spiral point for the flow on the unstable manifold. In the nonhyperbolic case, \( \alpha = 0 \), \( x^* \) has a two dimensional center manifold. The assumption (H5) that \( x^* \) is unstable implies that \( x^* \) is an unstable spiral point on the center manifold. The center manifold belongs to \( A \) since it consists of orbits which have compact closure in \( D \). \[\]

We denote by \( P \) the set of all nontrivial period orbits of (1.1) in \( D \).

**Proposition 2.4.** Let \( \gamma \in P \) and \( m_1 \) and \( m_2 \) be the eigenvalues of the Jacobian of a Poincaré map associated with a transversal section to the flow of (1.1) at a point \( q \) of \( \gamma \). Then exactly one of the following holds:

(i) \( \gamma \) is a saddle: \( 0 < m_1 < 1 < m_2 \) and the local unstable manifold of \( \gamma \), \( W^u_{loc}(\gamma) \), is homeomorphic to an annulus and is contained in \( A \).

(ii) \( \gamma \) is degenerate: \( 0 < m_1 = m_2 \) and a local center manifold of \( \gamma \), \( W^c(\gamma) \), exists which is homeomorphic to an annulus. The orbit \( \gamma \) separates the center manifold into two components. On at least one such component, the
flow of (1.1) spirals away from $\gamma$ and this component belongs to $A$. For the flow on the other component there are three possibilities: (a) the same as the above; (b) there are periodic orbits distinct from $\gamma$ on the component which converge to $\gamma$ in the Hausdorff metric, in which case the component belongs to $A$; (c) orbits spiral toward $\gamma$ on the manifold and in this case, we cannot assert that the component is contained in $A$. In all cases, $\gamma$ is unstable for the flow restricted to $W^u(\gamma)$.

Proof. By [13, Lemma 2.1], $\gamma$ has a simple Floquet multiplier $m_1$ satisfying $0 < m_1 < 1$ and all other multipliers exceed $m_1$ in modulus. Let the multipliers corresponding to the variational equation about the solution generating $\gamma$ consist of $m_1$, $m_2$, and one. Thus, $0 < m_1 < 1$ and $m_1 > m_2$. The assumption (H5) implies that $m_2 \geq 1$. The two cases described in (i) and (ii) are the only alternatives. The assertions concerning the unstable manifold of $\gamma$ are immediate in case (i). The unstable manifold is an annulus because the Poincaré map is orientation preserving on the unstable manifold ($m_2 > 0$). In the degenerate case, there exists a two dimensional center manifold for $\gamma$. The flow of (1.1), restricted to the center manifold, must be consistent with our assumption (H5) that $\gamma$ is unstable. This implies that $\gamma$ must be unstable for the flow of (1.1), restricted to the center manifold [9]. The possibilities are more easily described for the center manifold corresponding to the Poincaré map on a surface of section to $\gamma$ at the point $q$ of $\gamma$. This manifold is one dimensional and is separated into two components by deleting the fixed point $q$. By shrinking a component, if necessary, there are three cases: (1) The action of the map is to move all points away from $q$, or (2) the map moves all points toward $q$, or (3) there are fixed points of the map distinct from $q$ which converge to $q$. Since the stability of $\gamma$ is determined by the stability of the map on the center manifold, it must be the case that on at least one component, alternative (1) holds. The action of the map on the other component can satisfy either (1), (2), or (3). If either (1) or (3) holds for one component, then it must belong to $A$. Finally, the two dimensional center manifold of $\gamma$ is topologically an annulus since the Poincaré map, restricted to the one dimensional center manifold of the fixed point $q$, is orientation preserving.

Proposition 2.5. For each $x$ in $A - \{x^*\}$, $\omega(x)$ is an element of $P$ and $\alpha(x)$ is either $x^*$ or an element in $P$. In particular, $A$ consists of $x^*$, a nonempty set $P$ of periodic orbits, and a nonempty set of orbits which do not belong to $P$ and which have an element of $P$ as positive limit set and either $x^*$ or an element of $P$ as negative limit set.

Proof. This is easy to see if we look at the projected flow on $\Pi(A)$. By Proposition 2.3, $\Pi(x^*)$ is an unstable spiral point on $\Pi(A)$ since the unstable manifold of $x^*$, in the case in which $x^*$ is a hyperbolic saddle, or
the center manifold of $x^*$, in the case in which $x^*$ is a nonhyperbolic saddle, belongs to $A$. Hence $\Pi(x^*)$ cannot belong to the positive limit set of any point of $\Pi(A)$ other than itself. If $x \in A - \{x^*\}$ then $\omega(\Pi(x))$ belongs to $\Pi(A)$ and so it contains no equilibrium points. It follows that $\omega(\Pi(x))$ is $\Pi(\gamma)$ for some periodic orbit $\gamma \in P$, by the Poincaré–Bendixson theorem.

**Definition.** Let $\gamma_1, \gamma_2$ be distinct elements of $P$. Define $\gamma_1 \prec \gamma_2$, if the projection of $\gamma_1$ on $\mathbb{R}^2$, $\Pi(\gamma_1)$, is contained in the bounded open component of $\mathbb{R}^2 - \Pi(\gamma_2)$. Define $\gamma_1 \preceq \gamma_2$ if either $\gamma_1 = \gamma_2$ or $\gamma_1 \prec \gamma_2$. Obviously, the relation $\preceq$ on $P$ is a partial order relation.

**Definition.** Let $\gamma$ be an element of $P$. $\Pi(\gamma)$ is a Jordan curve on $\mathbb{R}^2$ which divides $\mathbb{R}^2$ into two open regions, one of which is bounded, the other unbounded. We use the following notations to denote the two components:

(i) $(\Pi(\gamma))_{\text{int}}$ = the bounded component of $\mathbb{R}^2 - \Pi(\gamma)$.
(ii) $(\Pi(\gamma))_{\text{ext}}$ = the unbounded component of $\mathbb{R}^2 - \Pi(\gamma)$.

We now define two subsets of $P$.

**Definition.** Let $P_i$ and $P_e$ be defined as follows:

(a) $\gamma$ is said to belong to $P_i$ if $\Pi(A)$ contains a neighborhood of $\Pi(\gamma)$ in $\Pi(\gamma) \cup (\Pi(\gamma))_{\text{int}}$ and $\Pi(\gamma)$ is unstable from its interior for the projected flow in $\mathbb{R}^2$ in the sense that the negative limit set of any point of $(\Pi(\gamma))_{\text{int}}$ is close to $\Pi(\gamma)$.

(b) $\gamma$ is said to belong to $P_e$ if $\Pi(A)$ contains a neighborhood of $\Pi(\gamma)$ in $\Pi(\gamma) \cup (\Pi(\gamma))_{\text{ext}}$ and $\Pi(\gamma)$ is unstable from its exterior for the projected flow in $\mathbb{R}^2$ in the same sense as in (a) except that $(\Pi(\gamma))_{\text{ext}}$ replaces $(\Pi(\gamma))_{\text{int}}$.

By Propositions 2.2 and 2.4, $P = P_i \cup P_e$.

**Proposition 2.6.** $P_i$ is empty.

**Proof.** Suppose not. Let $\gamma_0$ belong to $P_i$. Let $P_0 = \{\gamma \in P_i : \gamma \preceq \gamma_0\}$. As $(P_0, \preceq)$ is a partially ordered set, by Zorn's lemma there exists a maximal totally ordered subset $S$ of $P_0$ which contains $\gamma_0$. $S$ cannot contain a minimal element since if it did, then consider an orbit of the projected flow in the interior of this minimal element which has the minimal element as its negative limit set. The positive limit set of this orbit belongs to $\Pi(A)$ and cannot be $\Pi(x^*)$ since it is an unstable spiral point for the projected flow. So the positive limit set must be $\Pi(\gamma)$ for some $\gamma$ belonging to $P_0$ lying interior to the minimal element, contradicting the minimality of our minimal
element. Thus, for every $\gamma$ belonging to $S$ there exists $\gamma'$ belonging to $S$ such that $\gamma' < \gamma$.

By considering the positive limit set of a point of the nonempty set $F = \bigcap \{ [\Pi(\gamma) \cup (\Pi(\gamma'))_{\text{int}}] \cap \Pi(A) : \gamma \in S \}$, it is easy to show that $R = \{ \gamma \in P : \gamma \leq \gamma' \text{ for all } \gamma' \in S \}$ is nonempty. Note that $F$ cannot be a singleton since then it would necessarily be $\Pi(x^*)$, but $\Pi(x^*)$ is an unstable spiral point for the projected flow and hence is isolated from the projection of elements of $P$. Since every element of $P$ belongs to either $P_1$ or $P_2$, and $S$ does not contain any minimal element, it follows that $R \subseteq P_2$.

Let $U$ be a maximal totally ordered subset of $R$ with $\gamma_u \geq \gamma$, for all $\gamma \in U$. Since $\gamma_u$ belongs to $P_2$, there exists an orbit of the projected flow in $\Pi(A) \cap (\Pi(\gamma_u))_{\text{ext}}$, which has $\Pi(\gamma_u)$ as its negative limit set and $\Pi(\gamma_u)$ as its positive limit set for some $\gamma_u$ belonging to $P$. There are two cases depending on whether this heteroclinic orbit belongs to $(\Pi(\gamma_u))_{\text{int}}$ or to $(\Pi(\gamma_u))_{\text{ext}}$. In the first case, $\gamma_u < \gamma_u'$, so since $\gamma_u$ must belong to $R$ this contradicts the maximality of $\gamma_u$. In the second case, $\gamma_u$ must belong to $P_3$ by (H5) and the fact that $\gamma_u$ is the positive limit set of an orbit belonging to $(\Pi(\gamma_u))_{\text{ext}}$. But $\gamma_u$ also belongs to $R$ and these two facts imply that $S$ has a minimal element $\gamma'_u$ belonging to $(\Pi(\gamma_u))_{\text{int}}$, which is a contradiction to assertions above. Thus, $U$ does not contain a maximal element, i.e., for every $\gamma \in U$ there exists $\gamma'$ in $U$ such that $\gamma < \gamma'$.

By [1], $\bigcup \{ (\Pi(\gamma))_{\text{int}} : \gamma \in U \}$ is a 2-cell. Its boundary $C$ is an invariant set for the projected flow belonging to $\Pi(A)$ and it is contained in $(\Pi(\gamma))_{\text{int}}$ for each $\gamma$ belonging to $S$. As $\Pi(x^*)$ is an unstable spiral point, it cannot belong to $C$, so $C$ must be $\Pi(\gamma)$ for some $\gamma_0$ belonging to $P$. Obviously, $\gamma_0$ cannot belong to $P_1$, so it must belong to $P_2$. But then $\gamma_0$ is a maximal element of $U$, contradicting our reasoning above.

Thus we have reached a contradiction to the case where $U$ contains a maximal element and to the case where it does not. This proves the proposition.

**Proof of Theorem 1.1.** Suppose (H5) holds. Let $P(x^*) = \{ \gamma \in P : \Pi(x^*) \subseteq (\Pi(\gamma))_{\text{int}} \}$. $P(x^*)$ is nonempty and contains a minimal element since $\Pi(x^*)$ is an unstable spiral point for the projected flow. Clearly, $P(x^*)$ is totally ordered by "$\leq$" and, by Proposition 2.6, is a subset of $P_3$. Arguing exactly as with the totally ordered set $U$ in the previous proof, using the compactness of $\Pi(A)$, we can see that $P(x^*)$ contains a maximal element, $\gamma_M$. As $\gamma_M$ belongs to $P_3$, consider an orbit in $\Pi(A) \cap (\Pi(\gamma_M))_{\text{ext}}$ that has $\gamma_M$ as its negative limit set. The positive limit set of this orbit must be $\Pi(\gamma)$ for some $\gamma$ belonging to $P_2$ by Propositions 2.5 and 2.6. But then $\gamma > \gamma_M$, contradicting the fact that $\gamma_M$ is the maximal element of $P(x^*)$. This final contradiction to (H5) completes the proof of the theorem.

We no longer assume that (H5) holds.
Proof of Theorem 1.2. By Theorem 1.1, there exists an orbitally stable periodic orbit. We claim that every periodic orbit \( \gamma \) of (1.1) is isolated in the sense that there is a neighborhood of it containing no point of any other periodic orbit. To prove our claim, suppose that there exists a non-trivial periodic orbit \( \gamma_0 \) such that it is not isolated. Then there exists a sequence of periodic orbits \( \{\gamma_n\} \) of (1.1) and \( q_n \in \gamma_n \) such that \( \text{Dist}(q_n, \gamma_0) \to 0 \) as \( n \to +\infty \); here \( \text{Dist}(\cdot, \cdot) \) denotes the Euclidean distance in \( \mathbb{R}^3 \). \( \Pi(\gamma_0) \) is a nonisolated periodic orbit in \( \Pi(A) \) and \( \Pi(q_n) \) belongs to the periodic orbit \( \Pi(\gamma_0) \) in the plane and \( \text{Dist}(\Pi(q_n), \Pi(\gamma_0)) \to 0 \) as \( n \to +\infty \). But by the properties of planar systems, the periodic orbit \( \Pi(\gamma_0) \) must tend to \( \Pi(\gamma_0) \) in the Hausdorff metric as \( n \to \infty \) and the period of \( \Pi(\gamma_n) \) must tend to that of \( \Pi(\gamma_0) \). By the continuity of \( \Pi^{-1}|_{\Pi(A)} \), \( \gamma_n \) tends to \( \gamma_0 \) in the Hausdorff metric in \( \mathbb{R}^3 \) as \( n \to +\infty \). Furthermore, as \( \Pi|_A \) maps trajectories to trajectories, respecting parameterization (see [3], proof of Theorem 3.1), the period of \( \gamma_n \) must be the same as the period of \( \Pi(\gamma_0) \) which approaches the period of \( \gamma_0 \) as \( n \to +\infty \).

Let \( H \) be a plane in \( \mathbb{R}^3 \) passing through a point \( q \in \gamma_0 \) and orthogonal to \( f(q) \). Let \( Q \) denote the Poincaré, first-return, map defined in a neighborhood of \( q \) on \( H \). Then \( Q \) is analytic and \( q \) is a nonisolated fixed point of \( Q \) by the arguments of the previous paragraph. It follows that \( \gamma \) is degenerate and \( DQ(q) \) has eigenvalues \( m_1, m_2 \) with \( 0 < m_1 < 1 = m_2 \). Periodic orbits near \( \gamma_0 \) are in one-to-one correspondence with fixed points of \( Q \) in a neighborhood of \( q \), or equivalently, with solutions of \( F(x) = Q(x) - x = 0 \) for \( x \) near \( q \). By an affine change of variables we may assume that \( q = 0 \) and

\[
DF(0) = \begin{pmatrix} m_1 - 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

The Implicit Function Theorem implies that we can solve \( F_1(x_1, x_2) = 0 \) for \( x_1 = g(x_2), g(0) = 0 \), where \( g \) is analytic in a neighborhood of \( x_2 = 0 \). Thus \( F(x) = (F_1(x_1, x_2), F_2(x_1, x_2)) = 0 \) near \( x = 0 \) if and only if \( x = (g(x_2), x_2) \) and \( F_2(g(x_2), x_2) = 0 \) for \( x_2 \) near zero. As \( x = 0 \) is a nonisolated solution of \( F(x) = 0 \), \( x_2 = 0 \) is a nonisolated solution for the analytic map \( x_2 \mapsto F_2(g(x_2), x_2) \), and hence the latter map must vanish identically in a neighborhood \( I \) of \( x_2 = 0 \). It follows that each point of the analytic arc \( \Gamma = \{ (g(x_2), x_2) : x_2 \in I \} \) and is a fixed point of \( Q \). Each point of the arc \( \Gamma \) represents a distinct periodic orbit. \( M = \bigcup \{ \Pi(\gamma) : \gamma \text{ is a periodic orbit through some point on } \Gamma \} \) is an annulus in \( \mathbb{R}^2 \). By shrinking \( \Gamma \) if necessary, \( M \) is an open annulus in \( \mathbb{R}^2 \), through each point of which is a periodic orbit \( \Pi(\gamma) \) for some periodic orbit \( \gamma \) of (1.1). And for every pair of distinct periodic orbits in \( M \), \( \Pi(\gamma_1), \Pi(\gamma_2) \), either \( \gamma_1 < \gamma_2 \) or \( \gamma_2 < \gamma_1 \). If \( \Pi(\gamma) \) is a periodic orbit in \( M \) then \( \gamma \) is orbitally stable but not orbitally asymptotically stable for (1.1).
Let $S = \{ \gamma : \gamma$ is a nonisolated periodic orbit and $\gamma \geq \gamma_0$ or $\gamma \leq \gamma_0 \}$. $S$ is nonempty because $\gamma_0 \in S$. Define $G = \cup \{ \Pi(\gamma) : \gamma \in S \}$; then $G \subset \Pi(A)$ and $G$ is an open set in $\mathbb{R}^2$. In fact, for each $\gamma \in S$, the same arguments as those above show that there is an open annulus $M$, containing $\Pi(\gamma)$ in $\Pi(A)$ which consists of the projections of periodic orbits of (1.1). Therefore $G$ is a union of open annuli contained in $\Pi(A)$.

Let $W = \{ \Pi(\gamma) : \gamma > \gamma_0, \ \gamma \in S \text{ and } (\Pi(\gamma))_{\text{int}} \cap (\Pi(\gamma_0))_{\text{ext}} \subset G \}$. Then $W$ is not empty because $\Pi(\gamma) \in W$ for each $\Pi(\gamma) \subset M$. And for every pair $\Pi(\gamma_1)$, $\Pi(\gamma_2)$ belonging to $W$, either $\gamma_1 > \gamma_2 > \gamma_0$ or $\gamma_2 > \gamma_1 > \gamma_0$. Thus $W$ is totally ordered with respect to the obvious ordering on the set of all periodic orbits $\Pi P$.

For any $\Pi(\gamma)$ belonging to $W$, $(\Pi(\gamma))_{\text{int}}$ is an open 2-cell, so by [1], $R = \cup \{ ((\Pi(\gamma))_{\text{int}} : \Pi(\gamma) \in W \}$ is also an open 2-cell. Its boundary, $\partial R$, is contained in $\Pi(A)$ and it is a nonempty compact invariant set of (1.1). $\partial R$ cannot be a singleton $\Pi(x^*)$ and it does not contain any homoclinic orbits connecting $\Pi(x^*)$ to itself because $\Pi(x^*)$ is an unstable spiral point in $\mathbb{R}^2$. By the Poincaré–Bendixson theorem, there exists at least one periodic orbit $\Pi(\gamma_1)$ in $\partial R$ for some periodic orbit $\gamma_1$ in $A$. In fact, $\partial R = \Pi(\gamma_1)$ is the only possibility since $\partial R$ is homeomorphic to the unit circle in $\mathbb{R}^2$. That is, $\partial R$ consists solely of the periodic orbit $\Pi(\gamma_1)$. It follows that

$$(\Pi(\gamma_1))_{\text{int}} \cap (\Pi(\gamma_0))_{\text{ext}}$$

$$= \cup \{ \{(\Pi(\gamma))_{\text{int}} : \Pi(\gamma) \in W \} \cap (\Pi(\gamma_0))_{\text{ext}}$$

$$= \cup \{ \{(\Pi(\gamma))_{\text{int}} \cap (\Pi(\gamma_0))_{\text{ext}} : \Pi(\gamma) \in W \} \subset G.$$ 

Furthermore, $\gamma_1 \geq \gamma$ for all $\gamma \in P$ such that $\Pi(\gamma) \in W$.

Obviously $\gamma_1$ is not isolated, so repeating the arguments used above to show that $\Pi(\gamma_0)$ belongs to an annulus foliated by periodic orbits, one can get an open annulus $M_1$ containing $\Pi(\gamma_1)$ in $\Pi(A)$ consisting of the projections of periodic orbits of (1.1). But this implies that $\gamma_1 \in S$ and $\Pi(\gamma_1) \subset W$ and there exists $\Pi(\gamma^*) \subset M_1$ such that $\gamma^* > \gamma_1$, $\Pi(\gamma^*) \in W$, a contradiction. The claim is proved.

Returning to the proof of the theorem, suppose that the number of periodic orbits of (1.1) is infinite. Let $\{\gamma_m\}$ be an infinite sequence of distinct periodic orbits of (1.1) and let $\Omega = \{ y \in D : \lim_{k \to +\infty} x_{m_k} = y \text{ with } x_{m_k} \in \gamma_{m_k} \}$. Then $\Omega \subset A$ and it is nonempty and compact. Let $y \in \Omega$ and $x_{m_k} \in \gamma_{m_k}$, where $\lim_{k \to +\infty} x_{m_k} = y$. It follows that $x(t, x_{m_k}) \to x(t, y)$ uniformly on compact intervals. In particular, $\Omega$ is invariant. $\Omega$ cannot contain $x^*$ since $x^*$ is isolated from nontrivial periodic orbits. By Proposition 2.5, $\Omega$ contains at least one periodic orbit of (1.1), say $\gamma'$. But this will imply that $\gamma'$ is not isolated from the periodic orbits of (1.1), a contradiction to the previous claim. Thus the set of periodic orbits must be finite.
To prove the last assertion of the theorem, suppose that $\gamma$ is an orbitally stable periodic orbit of (1.1). Then $\gamma$ is isolated from other periodic orbits. Let $U$ be a neighborhood of $\gamma$ whose closure, $\text{clos}(U)$, contains no point of any other periodic orbit including $x^*$. Let $V \subset U$ be a neighborhood of $\gamma$ such that if $x \in V$, then $\phi^n(x) \subset U$. Then $\omega(x) \subset \text{clos}(U) \cap A$, and by [3, 6, 13], $\omega(x)$ must be a nontrivial periodic orbit of (1.1) for each $x \in V$. Thus $\gamma$ is orbitally asymptotically stable. The proof is complete.

**Proof of Theorem 1.3.** Under the hyperbolicity assumption it is clear that either $x^*$ is asymptotically stable or it is unstable, with a two dimensional unstable manifold, and in this case the orbitally stable periodic orbit, the existence of which is implied by Theorem 1.1, must be orbitally asymptotically stable. As each periodic orbit $\gamma$ is hyperbolic, it is either orbitally asymptotically stable or unstable with a two-dimensional unstable manifold belonging to $A$. In the latter case, the unstable manifold contains a neighborhood of $\gamma$ in the relative topology of $A$ by Proposition 2.2. In either case, $\gamma$ is an isolated periodic orbit in the sense that there is a neighborhood of $\gamma$ containing no point of any other periodic orbit, including $x^*$. A similar argument implies that there is a neighborhood of $x^*$ containing no point of any periodic orbit.

If the set $P$ is infinite then we may choose a countably infinite set $\gamma_n \in P$ and $x_n \in \gamma_n$ such that $x_n \to x$. In fact, each point $y \in \omega(x)$ is also a limit of a sequence $y_n \in \gamma_n$. But $w(x)$ either contains $x^*$ or is an element of $P$ by [6, Thm. 1] and this contradicts our earlier assertion that $x^*$ contains all elements of $P$ unless they are isolated from points of other elements of $P$. This contradiction establishes the finiteness of $P$.

In Proposition 2.4 we noted that if $\gamma \in P$ is hyperbolic and unstable then the eigenvalues $m_1$ and $m_2$ of the Jacobian of a Poincaré map associated with a transversal section to the flow of (1.1) at a point $q$ of $\gamma$ satisfy $0 < m_1 < 1 < m_2$. In [13, Thm. 1.3] it is shown that in this case the stable manifold $W^s(\gamma)$ is a two dimensional cylinder-like manifold foliated by strongly increasing curves. Thus each point $p$ of $W^s(\gamma) - \gamma$ satisfies $p \ll q$ or $q \ll p$ for some point of $q \in \gamma$. In particular, as $A$ is unordered, $A \cap (W^s(\gamma) - \gamma) = \emptyset$.

Let $\gamma$ and $\gamma'$ be not necessarily distinct periodic orbits and suppose $p \in W^s(\gamma') \cap W^s(\gamma)$ but $p$ does not belong to either $\gamma$ or $\gamma'$. If $\gamma$ is unstable then by the previous paragraph $p \notin A$, which is a contradiction to the fact that $W^u(\gamma') \subset A$. Thus $\gamma$ must be stable and the hyperbolicity of $\gamma$ implies that $W^s(\gamma)$ is an open set. Obviously then $W^s(\gamma)$ and $W^s(\gamma')$ intersect transversally at $p$. This establishes that $W^s(\gamma)$ and $W^s(\gamma')$ intersect transversally. The same argument applies if $\gamma' = x^*$.

Recall that either $x^*$ is asymptotically stable or $x^*$ is unstable with a two dimensional unstable manifold $W^u(x^*) \subset A$ and a one dimensional stable
manifold, $W^s(x^*)$, which consists of a totally ordered set of points [12, Thm. 2.8]. As $A$ is unordered, it follows that $W^s(x^*) \cap W^u(x^*) = \{x^*\}$. If $x^*$ is unstable, the same argument implies that $W^s(x^*) \cap W^u(y) = \emptyset$ for any $y \in A$.

The results of the previous two paragraphs imply (i) of the Morse–Smale conditions. In order to prove that (ii) holds we could appeal to [6, Thms. 2, 3] but a simpler argument is possible in our special situation. By [6, Thm. 1], every compact limit set in $D$ which does not contain $x^*$ is a periodic orbit. Thus, we need only show that every limit set containing $x^*$ is $\{x^*\}$. If not, that is, if $x^*$ belongs to a limit set $L$ but $\{x^*\} \neq L$, then $x^*$ is not asymptotically stable and therefore $W^s(x^*)$ is one dimensional and totally ordered. Furthermore, by the Butler–McGehee lemma [16], $L$ contains a point $p$ of $W^s(x^*) - x^*$. But $L \subset A$ so $A$ contains two distinct ordered points, namely $x^*$ and $p$. This contradiction to Proposition 2.1 proves that (ii) holds. Thus, we have established that the Morse–Smale conditions hold.

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