Global Stability for Mixed Monotone Systems

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We show that the embedding method described in [4, 8] leads immediately to the global stability results in [7]. It also allows extension of some results on global stability for higher order difference equations due to Gerry Ladas and collaborators. Further, we provide a new result which suggests that embedding into monotone systems may not be necessary for global stability results.

Keywords: mixed monotone system, monotone dynamical system, global stability

This paper is dedicated to Gerry Ladas on the occasion of his 70th Birthday

1. Introduction

The idea of embedding a dynamical system, whose generator has both increasing and decreasing monotonicity properties (positive and negative feedback), into a larger symmetric monotone dynamical system and exploiting the convergence properties of the latter is very old. For a discussion of history of the method, see [4, 8]. The method is repeatedly rediscovered and its implications are often underestimated. In this paper, we review the main results of the embedding method following [8] and then we show how it leads immediately to an improved version of a nice result on global stability due to Kulenović and Merino [7] for componentwise monotone maps that leave invariant a hypercube in Euclidean space. We then ask whether embedding a system into a larger monotone system is really necessary to obtain global stability results. On the face of it, it seems unnatural to pass to a larger dimensional dynamical system in order to gain information on the dynamics of a smaller one. We show that for the class of mixed-monotone systems, one can obtain global stability results directly without the need of embedding.

As noted in [8], the embedding method leads to a nice generalization of some results of Kulenović, Ladas and Sizer [5], also contained in the monograph of Kulenović and Ladas [6], on higher order difference equations with componentwise monotonicity.

2. Review of the Embedding Method

Let $X$ be an ordered metric space with closed order relation $\leq$. The closedness of the order relation means that if $x_n \leq y_n$ and $x_n \to x$, $y_n \to y$, then $x \leq y$. If $x \leq y$, define the order interval $[x, y] := \{z \in X : x \leq z \leq y\}$. Let $F : X \to X$ be continuous. Our focus is the discrete-time dynamical system defined by

$$x' = F(x)$$

where $x'$ denotes the successor to $x$.

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We say $F$ is mixed-monotone if there exists a continuous map $f : X \times X \to X$ satisfying:

1. $F(x) = f(x, x)$, $x \in X$.
2. $\forall y \in X, x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y)$.
3. $\forall x \in X, y_1 \leq y_2 \Rightarrow f(x, y_2) \leq f(x, y_1)$.

In short, $f$ is nondecreasing in its first variable and nonincreasing in its second. Roughly, $F$ is a map that combines both positive and negative feedback. We write $F^n(x)$ for the $n$-fold composition of $F$ acting on $x$. The omega limit set of a subset $A \subset X$ is denoted by $\omega_F(A)$ and that of a single point $x \in X$, is denoted by $\omega_F(x)$.

As shown in [4], (1) can be embedded in the symmetric discrete-time dynamical system

$$
x' = f(x, y) \\
y' = f(y, x)
$$

on $X \times X$. We use the notation $z = (x, y)$ and define

$$G(z) = G(x, y) = (f(x, y), f(y, x))$$

$G$ is called the symmetric map. Obviously, the diagonal

$$D = \{(x, x) : x \in X\}$$

is positively invariant under (2) and $G(x, x) = (F(x), F(x))$. The symmetry of $G$ can be expressed by defining the reflection operator $R(x, y) = (y, x)$ and observing that $G \circ R = R \circ G$.

The “southeast” ordering on $X^2 := X \times X$ is the closed partial order relation defined by

$$(x, y) \leq_C (\bar{x}, \bar{y}) \iff x \leq \bar{x} \text{ and } \bar{y} \leq y.$$ 

It’s name derives from the fact that the bigger point lies southeast of the smaller one. Note that $R : X^2 \to X^2$ is order reversing.

Although the map $F$ need not be monotone, the symmetric map $G$ is monotone.

**Lemma 2.1:** $G$ is monotone with respect to $\leq_C$ on $X \times X$ in the sense that

$$z \leq_C \bar{z} \implies G(z) \leq_C G(\bar{z}).$$

Moreover, the “above-diagonal set” $\{(x, y) \in X \times X : x \leq y\}$ is positively invariant under $G$.

The following result, proved in [8] (see also [2]), is a sharpened version of Theorem 7 in [4].

**Theorem 2.2:** Suppose that:

$$\exists x_0, y_0, \ x_0 \leq y_0, \text{ satisfying } f(x_0, y_0) \geq x_0, \ f(y_0, x_0) \leq y_0. \quad (3)$$

Then $F([x_0, y_0]) \subset [x_0, y_0]$ and for $z_0 = (x_0, y_0)$ and $n \geq 1$, we have:

$$z_0 \leq_C G^n(z_0) \leq_C G^{n+1}(z_0) \leq_C Rz_0.$$

Assume, in addition, that the monotone orbit

$$\{G^n(x_0, y_0)\}_{n \geq 1}$$

converges in $X$. 

$$\quad (4)$$
Then:

(i) there exist \( x^\ast, y^\ast \in [x_0, y_0] \) with \( x^\ast \leq y^\ast \) satisfying
\[
G^n(x_0, y_0) \longrightarrow (x^\ast, y^\ast) = G(x^\ast, y^\ast),
\]
implies that \( f(x^\ast, y^\ast) = x^\ast \), \( f(y^\ast, x^\ast) = y^\ast \).

(ii) If \( x \in [x_0, y_0] \) and \( \{F^n(x)\}_{n \geq 1} \) has compact closure in \( X \), then
\[
\omega_F(x) \subset [x, y].
\]

(iii) If \( F([x_0, y_0]) \) has compact closure in \( X \), then \( \omega_F([x_0, y_0]) \neq \emptyset \), and
\[
\omega_F([x_0, y_0]) \subset [x^\ast, y^\ast].
\]

(iv) If
\[
a, b \in [x_0, y_0], \ a \leq b, \ f(a, b) = a, \ b = f(b, a) \Rightarrow a = b
\]
holds then \( x^\ast = y^\ast \) and \( F(x^\ast) = x^\ast \). In this case, if \( x \in [x_0, y_0] \) and \( \{F^n(x)\}_{n \geq 1} \) has compact closure
in \( X \), then \( \omega_F(x) = \{x^\ast\} \).

As noted in [8], the hypothesis (4) may be satisfied under a variety of hypotheses on either the space \( X \) or the map \( f \). If \( X \) is a finite dimensional ordered Banach space or if \( X = L^p(\Omega) \) is a Lebesgue space then order bounded sequences converge. It holds if \( f \) has compactness properties. See [8].

The proof of Theorem 2.2 exploits the fact that (3) is equivalent to:
\[
(x_0, y_0) \leq_C G(x_0, y_0)
\]
and so the monotone symmetric map \( G \) has a monotone increasing (relative to \( \leq_C \)) orbit \( (x_n, y_n) = G^n(x_0, y_0) \) which must remain in the “above-diagonal” set \( \{(x, y) \in X \times X : x \leq y\} \) by Lemma 2.1. By monotonicity of \( G \) and the fact that \( G = (F, F) \) on the diagonal, one concludes that for \( x \in [x_0, y_0] \) we have
\[
x_n \leq F^n(x) \leq y_n
\]
where \( x_n \not\to x^\ast \) and \( y_n \not\to y^\ast \).

Applications of Theorem 2.2 can be found in [8]. See also those in [7].

3. A result of Kulenović and Merino

We show that the main result of Kulenović and Merino [7] follows from Theorem 2.2.

Let \( m, M \in \mathbb{R}^k \) satisfy \( m \leq M \) and let \( F : [m, M] \rightarrow [m, M] \) be a continuous map. Kulenović and Merino call \( F \) coordinate-wise monotone (cw-monotone) if \( F_i \) is monotone in \( x_j \) on \( [m_j, M_j] \) for \( 1 \leq i, j \leq k \). For each \( i, K = \{1, 2, \cdots, k\} \) can be partitioned into two disjoint subsets as follows:
\[
I_i = \{j \in K : F_i \text{ is nondecreasing or constant in } x_j\}
\]
and
\[
D_i = \{j \in K : F_i \text{ is nonincreasing and nonconstant in } x_j\}
\]
Given vectors \( x, y \in [m, M] \) denote by \((x_I, y_{D_i})\) the vector \( z \in rmR^k \) with \( z_l = x_l \) for \( l \in I_i \) and \( z_l = y_l \) for \( l \in D_i \). Observe that \( z \in [m, M] \). Define \( f : [m, M] \times [m, M] \rightarrow R^k \) by

\[
f_i(x, y) = F(x_I, y_{D_i})
\]

Kulenović and Merino give an alternate but equivalent definition of \( f \) as follows. Define

\[
\sigma_{ij} = \begin{cases} 
1 & \text{if } F_i \text{ is nondecreasing or constant in } x_j \\
-1 & \text{if } F_i \text{ is nonincreasing and nonconstant in } x_j
\end{cases}
\]

Then for \( 1 \leq i \leq k \),

\[
f_i(x, y) = F_i \left( \frac{1 + \sigma_{i1}}{2} x_1 + \frac{1 - \sigma_{i1}}{2} y_1, \frac{1 + \sigma_{i2}}{2} x_2 + \frac{1 - \sigma_{i2}}{2} y_2, \ldots, \frac{1 + \sigma_{ik}}{2} x_k + \frac{1 - \sigma_{ik}}{2} y_k \right)
\]

Observe that

(a) \( f([m, M] \times [m, M]) \subset [m, M] \).
(b) \( F(x) = f(x, x), x \in [m, M] \).
(c) \( f \) is nondecreasing in \( x \) and nonincreasing in \( y \).

The following result extends Theorem 3 of Kulenović and Merino [7].

**Theorem 3.1**: Let \( F : [m, M] \rightarrow [m, M] \) be a continuous cw-monotone map. Assume that

\[
a, b \in [m, M], \ a \leq b, \ f(a, b) = a, \ b = f(b, a) \Rightarrow a = b
\]

holds. Then there exists \( x_* \in [m, M] \) such that \( \omega_F(x) = x_* \) for all \( x \in [m, M] \).

**Proof**: Since \( f : [m, M] \times [m, M] \rightarrow [m, M] \), it follows that \( f(m, M) \geq m \) and \( f(M, m) \leq M \). Hence hypothesis (3) of Theorem 2.2 holds with \( x_0 = m \) and \( y_0 = M \). Indeed, all hypotheses of that theorem hold so the result follows from part (iv). \( \Box \)

Theorem 3.1 extends Theorem 3 in [7] because the additional restriction \( a \leq b \) appears in our premise (8) but not in the premise of II. of Theorem 3 of [7].

4. **To Embed or not to Embed**

One might wonder whether it is really necessary to embed (1) into the symmetric map (2) in order to obtain significant results. We begin by showing that a seemingly more powerful hypothesis than (8) is actually equivalent to it.

**Proposition 4.1**: Suppose that hypotheses (3) and (4) of Theorem 2.2 hold. Then hypothesis (8) holds if and only if

\[
a, b \in [x_0, y_0], \ a \leq b, \ f(a, b) \leq a, \ b \leq f(b, a) \Rightarrow a = b
\]

holds.

**Proof**: Suppose that (8) holds and that there exists \( a, b \in [x_0, y_0] \) such that \( a \leq b, \ f(a, b) \leq a, \ b \leq f(b, a) \).

In terms of the symmetric map \( G \), this means that \( G(a, b) \leq_C (a, b) \). Then \((x_0, y_0) \leq_C (a, b) \) and so by monotonicity of \( G \)

\[
(x_0, y_0) \leq_C G(x_0, y_0) \leq_C G(a, b) \leq_C (a, b).
\]
It follows that $G(x_+, y_+) = (x_+, y_+) = \lim_n G^n(x_0, y_0) \leq C (a, b)$. So $f(x_+, y_+) = x_+$, $f(y_+, x_+) = y_+$ and $x_+ \leq y_+$. (8) implies that $x_+ = y_+$. The inequality $(x_+, y_+) \leq C (a, b)$ implies that $b \leq y_+ = x_+ \leq a$ which, together with $a \leq b$ implies $a = b$. □

Now suppose that $X$ is a nonempty subset of an ordered metric space $Z$. We say that the continuous map $F : X \rightarrow X$ is weakly mixed-monotone if there exists a (not necessarily continuous) map $f : X \times X \rightarrow Z$ (note the range space is $Z$!) satisfying:

1. $F(x) = f(x, x)$, $x \in X$.
2. $\forall y \in X$, $x_1, x_2 \in X$, $x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y)$.
3. $\forall x \in X$, $y_1 \leq y_2 \Rightarrow f(x, y_2) \leq f(x, y_1)$.

**Theorem 4.2:** Suppose that $F$ is weakly mixed monotone and:

$$a, b \in X, a \leq b, f(a, b) \leq a, b \leq f(b, a) \Rightarrow a = b. \quad (10)$$

If $X$ contains the infimum and supremum of any pair of fixed points of $F$, then $F$ has at most one fixed point in $X$.

If $x \in X$ has the property that $\{F^t(x)\}_{t=0}^{\infty}$ is compact and contained in $X$ and such that $\inf \omega_F(x)$ and $\sup \omega_F(x)$ exist in $Z$ and belong to $X$, then $\omega_F(x)$ is a fixed point of $F$.

**Proof:** First, observe that (10) implies uniqueness of fixed points for $F$ in $X$. If $f(a, a) = a$ and $f(b, b) = b$ for $a, b \in X$, set $A = \inf \{a, b\}$ and $B = \sup \{a, b\}$. Then $A, B \in X$ by hypothesis and $A \leq a, b \leq B$. By the weak mixed monotone condition, $f(A, B) \leq f(a, a) = a$ and $f(A, B) \leq f(b, b) = b$ so $f(A, B) \leq A$. Similarly, $f(B, A) \geq B$. Therefore, $A = B$ and $a = b$ by (10).

Suppose that $x \in X$ has the property that $\{F^t(x)\}_{t=0}^{\infty}$ is compact and contained in $X$ and such that $c = \inf \omega_F(x)$ and $d = \sup \omega_F(x)$ exist in $Z$ and belong to $X$. Therefore $\omega_F(x)$ is nonempty and invariant: $F(\omega_F(x)) = \omega_F(x)$. If $y \in \omega_F(x)$, there exists $z \in \omega_F(x)$ such that $F(z) = y$. By the properties of $f$ and the fact that $c \leq z \leq d$, we conclude that $f(c, d) \leq y = F(z) = f(z, z) \leq f(d, c)$. Therefore, $f(c, d) \leq \omega(x) \leq f(d, c)$ which implies that $f(c, d) \leq c$ and $f(d, c) \geq d$. (10) implies that $c = d$. Therefore, $\omega_F(x)$ is a singleton, necessarily a fixed point. □

In the special case that $X = [x_0, y_0] \subset \mathbb{R}^k$ with any orthant ordering or $X = [x_0, y_0] \subset C(A, \mathbb{R}^k)$ where $A$ is a compact space and $C(A, \mathbb{R}^k)$ is ordered in the natural way from some orthant ordering on $\mathbb{R}^k$, and if orbits of $F$ are precompact, then the second part of Theorem 4.2 extends Theorem 2.2 (iv). Indeed, compact subsets of $C(A, \mathbb{R}^k)$ have infima and suprema.

Observe from the proof of Theorem 4.2 that the map $f$ is used in a very limited way compared to its use in Theorem 2.2. Unlike its use in the proof of Theorem 2.2 where it defines the map $G$, in the proof of Theorem 4.2 there is no need to iterate $f$, nor is its continuity required. These facts allow weakening the hypotheses on $f$ for Theorem 4.2.

**References**

