Big Monkey & Little Monkey compete for Fruit

Fruit: 10Kc energy, in tree.  
Big Monkey: costs 2 Kc to get fruit.  
Little Monkey: negligible cost to get fruit.

Both Monkeys wait: 0 Kc energy for both.

Both Monkeys climb: Big Monkey gets 7, Little Monkey gets 3.

Big Monkey climbs/Little Monkey waits: Big Monkey gets 6, Little Monkey gets 4.

Big Monkey waits/Little Monkey climbs: Big M. gets 9, Little M. gets 1.

assumption: each monkey seeks to maximize energy intake.
Game Tree if Big Monkey decides first.

Vocabulary: root node, branches, terminal nodes (leaf), action, strategy.

Big Monkeys strategies:
\[ w = \text{wait} \]
\[ c = \text{climb} \]

Little Monkeys strategies:
\[ ww = \text{wait regardless of what B. Monkey does} \]
\[ wc = \text{do what B. Monkey does} \]
\[ cw = \text{do opposite of what B Monkey does} \]
\[ cc = \text{climb no matter what B. Monkey does} \]
Normal Form if Big Monkey decides first.

L. Monkey

<table>
<thead>
<tr>
<th></th>
<th>cc</th>
<th>cw</th>
<th>wc</th>
<th>ww</th>
</tr>
</thead>
<tbody>
<tr>
<td>B. Monkey</td>
<td>w</td>
<td>9,1</td>
<td>9,1</td>
<td>0,0</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>5,3</td>
<td>4,4</td>
<td>5,3</td>
</tr>
</tbody>
</table>

Nash equilibria: strategy pair such that each is best response to other.
A row and column such that entry in the intersection is biggest in column for player 1 and biggest in row for player 2.
(w, cw), (w, cc), (c, ww).

(w, cc) is ”faulty” because strategy cw dominates strategy cc.
(c, ww) is not subgame perfect since it involves an incredible threat that L. Monkey will wait when B. Monkey waits.
Game Tree & Normal Form if both monkeys decide simultaneously.

<table>
<thead>
<tr>
<th>L. Monkey</th>
<th>c</th>
<th>w</th>
</tr>
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<tbody>
<tr>
<td>B. Monkey</td>
<td>5,3</td>
<td>4,4</td>
</tr>
<tr>
<td>w</td>
<td>9,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Nash Equilibria: \((w, c)\) with payoff 9, 1 and \((c, w)\) with payoff 4, 4.

Also a ”mixed strategy” where each monkey tosses fair coin and takes w if heads, c if tails!

Note: doted line signifies information set consisting of 2 Little Monkey nodes.
The Normal Form Game

Set of players \( i = 1, 2, \ldots, n \).
A set \( S_i \) of strategies for player \( i = 1, 2, \ldots, n \).
The set \( S \) of strategy profiles \( s = (s_1, s_2, \ldots, s_n) \) where \( s_i \in S_i \).
A payoff function \( \pi_i : S \rightarrow R \) for player \( i \).
\( \pi_i(s) \) = payoff to player \( i \) if players use strategy profile \( s = (s_1, s_2, \ldots, s_n) \).

\( s^* \) is **Nash equilibrium** if for every player \( i \):

\[
\pi_i(s^*) \geq \pi_i(s_1^*, s_2^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)
\]

for all \( s_i \in S_i \).

\( s_i^* \) is player \( i \)'s best response to

\( (s_1^*, s_2^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*) \).
Max-Min Strategy profile

Given \( s_i \in S_i \) define
\[
h_i(s_i) = \min \pi_i(s_1, s_2, \cdots, s_{i-1}, s_i, s_{i+1}, \cdots, s_n)\]
where the minimum is over all \( s_j \in S_j, j \neq i \). my opponents are going to "do me in"!
suppose that \( s_i^* \in S_i \) satisfies \( h_i(s_i^*) = \max h(s_i) \). where the maximum is over all \( s_i \in S_i \). \( s_i^* \) is the best player \( i \) can do if his opponents conspire against her.

\[
s^* = (s_1^*, s_2^*, \cdots, s_n^*)\]
is called a max-min strategy profile.

For B. Monkey (chooses first)-L. Monkey, \((c, cw), (c, cc)\) are max-min strategy profiles.
Definitions From Owen

A **strategy** for player $i$ is a function which assigns, to each of player $i$’s information sets $S^j_i$, one of the branches which follows a representative node of $S^j_i$.

Player $i$ has **perfect information** in a game if each of his information sets consist of only one node.

A game has **perfect information** if each player has perfect information.

Theorem 1.4.5 (pg 8, Owen) Every finite, $n$-person game with perfect information has at least one Nash equilibrium.
Symmetric Games

A 2-person game is **symmetric** if $S_1 = S_2$ (players have same strategy set) and if
\[
\pi_1(s, s') = \pi_2(s', s)
\]
for all $s, s'$ in $S_1$ (players get same payoff if they play same strategy against same opponent strategy).

Fact #1: $(s, s') \in A_1$ if and only if $(s', s) \in A_2$. (prove this!)

Fact #2: $(s, s')$ is Nash if and only if $(s', s)$ is Nash. (prove this!)

$(s, s')$ is a **symmetric Nash equilibrium** if $s = s'$.

It is natural to exclude nonsymmetric Nash equilibria $(s \neq s')$ from a symmetric game. The players are identical, have identical strategies and payoffs and have no idea if they are player 1 or player 2—which one is supposed to play $s$ and which one to play $s'$?

Prisoners Dilemma and the Hawk-Dove games are symmetric games.
Two-player games with mixed strategies

Let player 1 have \( m \) strategies labeled \( 1, 2, \cdots, m \) and player 2 have \( n \) strategies labeled \( 1, 2, \cdots, n \)

\( a_{ij} = \) payoff to player 1 if player 1 uses his \( i \)'th strategy against player 2’s \( j \)'th strategy.

\( b_{ij} = \) payoff to player 2 if player 1 uses his \( i \)'th strategy against player 2’s \( j \)'th strategy.

\( A = (a_{ij}) \) and \( B = (b_{ij}) \) are \( m \times n \) matrices.

\( x = (x_1, x_2, \cdots, x_m) \) is a player 1 “mixed strategy” if each \( x_i \geq 0 \) and \( \sum_{i=1}^{m} x_i = 1 \). It means ”play \( i \)'th strategy with probability \( x_i \)”. 

\((1, 0, 0, \cdots, 0)\) means play strategy 1 every time! This is just the first pure strategy. Likewise, \((0, 1, 0, \cdots, 0)\) is the 2nd pure strategy.
Payoffs in mixed strategy games

If player 1 plays his $i$ strategy against player 2 who plays mixed strategy $y = (y_1, y_2, \cdots, y_n)$ where $y_j \geq 0$ and $\sum_{j=1}^{n} y_j = 1$ then his expected payoff is:

$$\Pi_i = \sum_{j=1}^{n} a_{ij} y_j.$$ 

If player 1 plays with mixed strategy $x$ and player 2 uses $y$, then player 1 expected payoff is:

$$\pi_1(x, y) = \sum_{i=1}^{m} \Pi_i x_i = \sum_{i=1}^{m} x_i \left( \sum_{j=1}^{n} a_{ij} y_j \right) = xA y^T.$$ 

By similar reasoning, player 2 payoff is:

$$\pi_2(x, y) = xB y^T.$$ 

where
Nash equilibrium

Mixed strategy profile \((x^*, y^*)\) is Nash if:
\[
x A(y^*)^T \leq x^*A(y^*)^T \quad \text{for all player 1 strategies } x.
\]
("\(x^*\) is best response to \(y^*\).")

and

\[
x^*B y^T \leq x^*B(y^*)^T \quad \text{for all player 2 strategies } y.
\]
("\(y^*\) is best response to \(x^*\).")
Symmetric Games

Each player has same strategy set, numbered identically.

\( B = A^T \), the transpose of \( A \), because:

\( b_{ij} = \) payoff to player 2 if player 1 uses his \( i \)'th strategy against player 2’s \( j \)'th strategy.

\( = a_{ji} \), the payoff player 1 receives when playing \( j \)'th strategy against player 2’s \( i \)'th strategy.

\((x^*, x^*)\) is Nash if and only if

(1) \( xA(x^*)^T \leq x^*A(x^*)^T \) for all \( x \).

This is because:

\( x^*A^Ty^T \leq x^*A^T(x^*)^T \) follows immediately from (1) and the rule \((ABC)^T = C^TB^TA^T\).
"Fundamental Theorem"

\((x^*, y^*)\) is Nash if and only if:

(a) \([A(y^*)^T]_i = x^*A(y^*)^T\) for all \(i\) such that \(x^*_i > 0\). and

(b) \([A(y^*)^T]_i \leq x^*A(y^*)^T\) for all \(i\) such that \(x^*_i = 0\). and

(c) \([x^*B]_j = x^*B(y^*)^T\) for all \(j\) such that \(y^*_j > 0\). and

(d) \([x^*B]_j \leq x^*B(y^*)^T\) for all \(j\) such that \(y^*_j = 0\).

For symmetric game, \((x^*, x^*)\) is Nash \(\iff\) (a) & (b) hold with \(y^* = x^*\).
Useful Inequalities?

(1) $xA(y^*)^T \leq x^*A(y^*)^T$ for all player 1 strategies $x$.

holds if and only if:
(2a) $[A(y^*)^T]_i = x^*A(y^*)^T$ for all $i$ such that $x^*_i > 0$.  
\textbf{and}
(2b) $[A(y^*)^T]_i \leq x^*A(y^*)^T$ for all $i$ such that $x^*_i = 0$.

In fact, putting $x = (0, \cdots, 0, 1, 0, \cdots, 0)$, "1 in the $i$'th place", in the inequality (1) proves that (2b) holds for every $i$. 


(1) $\Rightarrow$ (2a)

If $[A(y^*)^T]_{i'} < x^*A(y^*)^T$ for some $i'$ such that $x_{i'}^* > 0$ then since $[A(y^*)^T]_i \leq x^*A(y^*)^T$ for all $i$, we have

$$x^*A(y^*)^T = \sum_i x_i^*[A(y^*)^T]_i$$

$$= \sum_{i \neq i'} x_i^*[A(y^*)^T]_i + x_{i'}^*[A(y^*)^T]_{i'}$$

$$< \sum_{i \neq i'} x_i^*[x^*A(y^*)^T] + x_{i'}^*[x^*A(y^*)^T]$$

$$= x^*A(y^*)^T \sum_i x_i^*$$

$$= x^*A(y^*)^T$$

A big fat contradiction! So (2a) must hold. It is easy to see that if (2a) and (2b) hold then (1) holds. Try it!
Completely mixed Symmetric Nash

By the “Fundamental theorem” \((x^*, x^*)\) is a completely mixed Nash if and only if
\([A(x^*)^T]_i = x^* A(x^*)^T\) for all \(i\). This means that

\[
\begin{align*}
\lambda &= a_{11}x^*_1 + a_{12}x^*_2 + \cdots + a_{1n}x^*_n \\
\lambda &= a_{21}x^*_1 + a_{22}x^*_2 + \cdots + a_{2n}x^*_n \\
&\quad \vdots \\
\lambda &= a_{n1}x^*_1 + a_{n2}x^*_2 + \cdots + a_{nn}x^*_n \\
1 &= x^*_1 + x^*_2 + \cdots + x^*_n
\end{align*}
\]

where \(\lambda = x^* A(x^*)^T\) is the payoff to each player.

This represents \(n + 1\) equations in the \(n + 1\) unknowns \(x^*_1, \ldots, x^*_n, \lambda\).
Completely mixed Asymmetric Nash

By the “Fundamental theorem” \((x^*, y^*)\) is a completely mixed Nash if and only if

\[
\lambda = a_{11}y_1^* + a_{12}y_2^* + \cdots + a_{1n}y_n^*
\]
\[
\lambda = a_{21}y_1^* + a_{22}y_2^* + \cdots + a_{2n}y_n^*
\]
\[
\vdots = \cdots
\]
\[
\lambda = a_{m1}y_1^* + a_{m2}y_2^* + \cdots + a_{mn}y_n^*
\]
\[
1 = y_1^* + y_2^* + \cdots + y_n^*
\]

and

\[
\mu = b_{11}x_1^* + b_{21}x_2^* + \cdots + b_{m1}x_m^*
\]
\[
\mu = b_{12}x_1^* + b_{22}x_2^* + \cdots + b_{m2}x_m^*
\]
\[
\vdots = \cdots
\]
\[
\mu = b_{1n}x_1^* + b_{2n}x_2^* + \cdots + b_{mn}x_m^*
\]
\[
1 = x_1^* + x_2^* + \cdots + x_m^*
\]

where \(\lambda = x^*A(y^*)^T\) and \(\mu = x^*B(y^*)^T\).
Maple code
Asymmetric Two-Player Game
applied to sec. 4.11 one-card two-round poker
with bluffing p.61
A: payoff to player one \( a_{ij} \) = payoff if player 1
uses i, player 2 uses j;
i : 1 = rrbb, 2 = rr /f; j : 1 = ss, 2 = sf, 3 = f;
with(linalg);
\[
A := \text{array}(1..3,1..3); \text{payoffs}
\]
\[
\]
player 2 payoff matrix. \( b_{ij} \) = payoff when player
1 uses i, player 2 uses j.
\[
B := \text{array}(1..3,1..3); \text{payoffs}
\]
\[
B[1, 1] := 0; B[1, 2] := -4; B[1, 3] := -2;
\]
now we create some vectors to augment the payoff matrices

\[ n := \text{array}(1..3, 1..1); \]
\[ n[1, 1] := -1; n[2, 1] := -1; n[3, 1] := -1; \]
\[ m := \text{array}(1..1, 1..4); \]
\[ m[1, 1] := 1; m[1, 2] := 1; m[1, 3] := 1; m[1, 4] := 0; \]
\[ r := \text{array}(1..4, 1..1); \]
\[ r[1, 1] := 0; r[2, 1] := 0; r[3, 1] := 0; r[4, 1] := 1; \]
we are ready to solve for player 1’s weights and player 2’s payoff

\[ C := \text{augment}(\text{transpose}(B), n); \]
\[ CW := \text{stackmatrix}(C, m); \]
\[ \text{linsolve}(CW, r); \]
we now solve for player 2’s weights and player 1’s payoff.

\[ C := \text{augment}(A, n); \]
\[ CW := \text{stackmatrix}(C, m); \]
\[ \text{linsolve}(CW, r); \]
Three-Agent Symmetric Games

Each player has same strategy set $S$. Symmetry: \( \pi_1(s, s', s'') = \pi_1(s, s'', s') \)
"Player 1 doesn’t care whose playing the opposing strategies".
And \( \pi_2(s, s', s'') = \pi_1(s', s'', s') \), \( \pi_3(s, s', s'') = \pi_1(s'', s, s') \)
"Players 2 and 3 get same payoffs as player 1". Conclusion: need only one payoff function \( \pi_1 = \pi \).

If \( S = \{1, 2, \cdots, n\} \) and notation: \( \pi_{ijk} = \pi(i, j, k) \)
Our player 1 payoff if he plays mixed strategy \( x \) against player 2 mixed strategy \( y \) and player 3 strategy \( z \) is:
\[
\pi(x, y, z) = \sum_{i,j,k=1}^{n} \pi_{ijk} x_i y_j z_k
\]
Symmetric Nash

\((x^*, x^*, x^*)\) is Nash if \(\pi(x, x^*, x^*) \leq \pi(x^*, x^*, x^*)\) for all \(x\).

It’s enough to make player 1 happy by symmetry!

Fundamental Theorem:

\[\pi(e_l, x^*, x^*) = \pi(x^*, x^*, x^*)\] for all \(l\) for which \(x^*_l > 0\).

\[\pi(e_m, x^*, x^*) \leq \pi(x^*, x^*, x^*)\] for \(m\) where \(x^*_l = 0\).

Equations for completely mixed Nash: (as usual \(\lambda = \pi(x^*, x^*, x^*)\))

\[
0 = \sum_{j,k=1}^{n} \pi_{ljk}x^*_jx^*_k - \lambda, \quad l = 1, 2, \cdots, n
\]

\[
1 = x^*_1 + x^*_2 + \cdots + x^*_n
\]

notice that for each \(l\), \((\pi_{ljk})_{j,k}\) is a symmetric \(n \times n\) matrix.
Completely mixed Symmetric Nash: $n = 3$

\[
0 = \pi_{l11}(x_1^*)^2 + \pi_{l22}(x_2^*)^2 + \pi_{l33}(x_3^*)^2 + 2\pi_{l12}x_1^*x_2^* + 2\pi_{l13}x_1^*x_3^* + 2\pi_{l23}x_2^*x_3^* - \lambda, \ l = 1, 2, 3
\]

\[
1 = x_1^* + x_2^* + x_3^*
\]
Evolutionarily Stable Strategy (ESS)

Consider a symmetric, 2-player, normal form game with strategy set $S = \{1, 2, \ldots, n\}$. Payoff matrix for player 1 is $A$; for player 2 is $A^T$.

Probability vector $x^*$ is ESS if $xA(x^*)^T \leq x^*A(x^*)^T$ and for all $x$ such that $xA(x^*)^T = x^*A(x^*)^T$ it follows that $x^*Ax^T > xA^Tx^T$.

Note if $x^*$ is ESS, then $(x^*, x^*)$ is Nash! This definition is equivalent to the following one:

$x^*$ is ESS if for all $x \neq x^*$, there exists $\epsilon_x > 0$ such that

$$x^*A[\epsilon x + (1 - \epsilon)x^*]^T > xA[\epsilon x + (1 - \epsilon)x^*]^T$$

for all $\epsilon$ satisfying $0 < \epsilon < \epsilon_x$. 

22
Useful Fact #1

If, for some $i$, $a_{ii} > a_{ji}$ for all $j \neq i$, then the pure strategy $e_i$ (play strategy $i$ always) is ESS. In words, $a_{ii}$ is the biggest entry in its column.

Proof: show $e_i A(e_i)^T > x A(e_i)^T$ for all $x \neq e_i$ so $e_i$ is the best response to itself!

\[ e_i A(e_i)^T - x A(e_i)^T = a_{ii} - \sum_{j=1}^{n} a_{ji} x_j \]

\[ = [ \sum_{j=1}^{n} x_j ] a_{ii} - \sum_{j=1}^{n} a_{ji} x_j \]

\[ = \sum_{j=1}^{n} x_j (a_{ii} - a_{ji}) \]

\[ = \sum_{j \neq i}^{n} x_j (a_{ii} - a_{ji}) > 0 \]

if $x_j > 0$ for some $j \neq i$. 

23
Useful Fact #2

If $x^*$ is ESS then there cannot exist a Nash $(x, x)$ with support of $x$ contained in support of $x^*$.

Recall, support of $x$ is $\{i : x_i > 0\}$, "the strategies used by $x". So there can be no Nash using a subset of the strategies used by $x^*$.

In particular, if $x^*$ is a completely mixed ESS, then there can be no other Nash!

If $(x^*, x^*)$ is a completely mixed Nash and if there are other Nash, then $x^*$ is not ESS!

See next page for proof.
Proof of Fact #2: \((x^*, x^*)\) is Nash so Fundamental Theorem says \([A(x^*)^T]_i = x^*A(x^*)^T\) for all \(i\) in support of \(x^*\). Write SPT for support of \(x^*\). Since support \(x\) is a subset of SPT, \(\sum_{i \in SPT} x_i = 1\) so multiplying the equation above by \(x_i\) and adding we get:

\[
\sum_{i=1}^{n} x_i[A(x^*)^T]_i = \sum_{i \in SPT} x_i[A(x^*)^T]_i
= x^*A(x^*)^T \sum_{i \in SPT} x_i
= x^*A(x^*)^T
\]

or \(xA(x^*)^T = x^*A(x^*)^T\). But then, since \(x^*\) is ESS:

\[
x^*A(x)^T > Ax^T
\]

which means \((x, x)\) is not Nash after all!
Correlated Equilibria

Let $A, B$ be $m \times n$ payoff matrices for a two-player, normal form game.

An $m \times n$ matrix $P = (p_{ij})$ is a **correlated mixed strategy** if $p_{ij} \geq 0$ and $\sum_{i,j} p_{ij} = 1$.

Interpretation: Player 1 receives signal to play her $i$th strategy and Player 2 simultaneously receives signal to play her $j$th strategy with probability $p_{ij}$.

$P$ is an **equilibrium correlated strategy** if, Player 1 cannot gain from disobeying her signal, assuming that Player 2 obeys her signal, and vice-versa.

useful formulas from probability:

\[
\text{Prob(Player 1 receives signal to play } i) = \sum_j p_{ij}
\]

\[
\text{Prob(P2 rec’d signal play } k/\text{ P1 rec’d signal play } i) = \frac{\text{Prob(P2 rec’d } k \text{ and P1 rec’d } i)}{\text{Prob(P1 rec’d } i)} = \frac{p_{ik}}{\sum_j p_{ij}}
\]
Correlated Equilibria Conditions

If P1 gets signal i, uses i, and P2 obeys signal, Expected Payoff to P1:

\[
= a_{i1} \text{Prob}(P2 \text{ gets } 1/ P1 \text{ got } i) \\
+ a_{i2} \text{Prob}(P2 \text{ gets } 2/ P1 \text{ got } i) + \cdots \\
= a_{i1} \left[ p_{i1} / \sum_j p_{ij} \right] + a_{i2} \left[ p_{i2} / \sum_j p_{ij} \right] + \cdots \\
= \sum_j a_{ij} p_{ij} / \sum_j p_{ij}
\]

If P1 gets signal i, uses k, and P2 obeys signal, Expected Payoff to P1:

\[
= a_{k1} \text{Prob}(P2 \text{ gets } 1/ P1 \text{ got } i) \\
+ a_{k2} \text{Prob}(P2 \text{ gets } 2/ P1 \text{ got } i) + \cdots \\
= a_{k1} \left[ p_{i1} / \sum_j p_{ij} \right] + a_{k2} \left[ p_{i2} / \sum_j p_{ij} \right] + \cdots \\
= \sum_j a_{kj} p_{ij} / \sum_j p_{ij}
\]
Correlated Equilibria Conditions continued

P1 can't do better by disobeying signal, given P2 obeys:

\[ \sum_j a_{ij} p_{ij} / \sum_j p_{ij} \geq \sum_j a_{kj} p_{ij} / \sum_j p_{ij} \]

or, \[ \sum_j (a_{ij} - a_{kj}) p_{ij} \geq 0 \]

For every \( k \) and \( i \)!

Similarly, P2 can’t do better by disobeying, given P1 obeys:

\[ \sum_i (b_{ij} - b_{il}) p_{ij} \geq 0 \]

for all \( l \) and \( j \).
Correlated Equilibria Conditions 2-by-2 case:

\[
\begin{align*}
(a_{11} - a_{21})p_{11} + (a_{12} - a_{22})p_{12} & \geq 0 \\
(a_{21} - a_{11})p_{21} + (a_{22} - a_{12})p_{22} & \geq 0 \\
(b_{12} - b_{11})p_{12} + (b_{22} - b_{21})p_{22} & \geq 0 \\
(b_{11} - b_{12})p_{11} + (b_{21} - b_{22})p_{21} & \geq 0 \\
p_{11} + p_{12} + p_{21} + p_{22} & = 1 \\
p_{ij} & \geq 0.
\end{align*}
\]