The course covers the basics of applied analysis including: modes of convergence (pointwise, uniform, and mean square) of sequences and series of functions with particular attention to Fourier series; metric space topology with emphasis on compactness and completeness and the Ascoli-Arzela Theorem; the Contraction Mapping Principle with application to the initial value problem for ordinary differential equations and the Implicit Function Theorem; A brief treatment of Lebesgue integration leading to $L^p$-spaces; elementary properties of orthonormal bases for Hilbert space; The Fourier transform and its elementary properties.

How does this course compare with the analysis course you may have had as an undergrad? A substantial fraction of the material of this course is contained in the text “Principles of Mathematical Analysis”, by W. Rudin, often used as a text for analysis courses at advanced undergraduate level. In particular, Chapters 2,4,7-9,11 of Rudin overlap substantially with the online text. We will cover Hilbert spaces and $L^p$-spaces which are not covered in Rudin. Students who have already completed an analysis course at the level of Rudin’s book may wish to take MAT570 in place of this course.

On Reserve at Noble Library: Principles of Mathematical Analysis, by W. Rudin
for topics in Lebesgue Integration I am using:
Real Analysis, Modern Techniques and Their Applications, G. Folland

Homework:
Chapter 1:
Exercise 1.6, 1.9, 1.12, 1.15, 1.20*
* use 1.20 show that:

1. If $A$ is a square matrix then $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ converges to a square matrix, called $\exp(A)$. $A^0 = I$, identity matrix. You may assume that $\|A\| = \max\{|Av| : |v| = 1\}$ is a norm on the vector space of square matrices. Here $|v|$ is a norm on vectors. Observe that $|Av| \leq \|A\||v|$ and this leads to the inequality $\|AB\| \leq \|A\|\|B\|$ for square matrices $A, B$.

2. If $A$ is a square matrix with $\|A\| < 1$ show that $\sum_{n=0}^{\infty} A^n$ converges to a square matrix $C$ and $C = (I - A)^{-1}$.

3. Show that the set of invertible matrices is an open subset of the $n \times n$ matrices, where the latter space is given the norm $\|A\| = \max_{ij} |a_{ij}|$. Hint: use properties of the determinant.

Chapter 2: 2.2, 2.3, 2.4, 2.5, 2.7
Use 1.20 to prove:

1. If $g_n \in C([a, b]), n \geq 1,$ and $M_n > 0$ satisfies $|g_n(x)| \leq M_n, a \leq x \leq b,$ and $\sum_{n=1}^{\infty} M_n < \infty$ show that $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly in $C([a, b])$ (to a continuous function, of course!).

3. If $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^p} \sin(nx)$, show that $f$ is continuously differentiable on $[0, 2\pi]$ if $p > 2$.

4. Let $f : [a, b] \times R \to R$ be continuous and suppose that $\{y_n(x)\}_{n=1}^{\infty}$ is a sequence of solutions of $y'(x) = f(x, y(x)), a \leq x \leq b$. If there exists $M > 0$ such that $|y_n(x)| \leq M, n \geq 1, a \leq x \leq b$, show that there is a subsequence $\{y_n\}$ which converges uniformly on $[a, b]$ to a function $y : [a, b] \to R$ which is a solution of the differential equation on $[a, b]$. Hint: $f$ is uniformly continuous on compact sets.

Chapter 3: 3.3, 3.4, 3.5, 3.7

Chapter 12: 12.7. Also do 8-11 below:

8. Let $(X, A)$ be a measurable space. If $f, g : X \to R$ are Borel measurable, show that $f + g$ is too. Hint: show $\{x \in X : f(x) + g(x) < c\} = \bigcup_{r \in Q} (\{x \in X : f(x) < r\} \cap \{x \in X : g(x) < c - r\}).$ $Q = \text{Rationals.}$

9. If $\phi$ and $\psi$ are non-negative simple functions, prove:

(a) if $c \geq 0$, $\int c\phi = c \int \phi$.

(b) $\int (\phi + \psi) = \int \phi + \int \psi$.
(c) if \( \phi \leq \psi \), then \( \int \phi \leq \int \psi \).

Hint for b,c: if \( \phi = \sum_i a_i \chi_{E_i} \) and \( \psi = \sum_j b_j \chi_{F_j} \) are in standard form, then \( E_i = \cup_j (E_i \cap F_j) \) is a disjoint (finite) union.

10. Compute and justify:
   a. \( \lim_{n \to \infty} \int_0^\infty (1 + \frac{x}{n})^{-n} \sin(\frac{x}{n})dx \)
   b. \( \lim_{n \to \infty} \int_0^1 x^{-1/2} \frac{nx}{1+nx} dx \)
   c. \( \lim_{n \to \infty} \int_0^1 x^n e^{x^2} dx \)

You may assume the Lebesgue integral agrees with Riemann integral when both are defined.

11. If \( f \) is bounded and Lebesgue measureable, show that

\[
g(s) = \int_0^\infty e^{-st} f(t) dt, \quad s > 0
\]

is continuous.

Chapter 12& 5: 12.16, 5.3, 5.5.

Also do the following:

1. Show that the closed unit ball in \( l^p(N) \) and in \( L^p([0,1]) \) is not compact. \( L^p([0,1]) \) is with respect to Lebesgue measure on \([0,1]\).

2. Let \( T : C^1([0,1]) \to C([0,1]) \) be defined by differentiation \( Tf = f' \). Show that \( T \) is a bounded linear transformation.

Chapter 6: 2,3,6,7,12.

Also do the following:

1. Let \( \{v_1, v_2, \cdots, v_n\} \) be an orthonormal basis for a subspace \( M \) of a Hilbert space \( H \). If \( x \in H \) show that the unique closest point \( y \in M \) to \( x \) is \( y = \sum_{k=1}^n (v_k, x)v_k \). Do not simply use Theorem 6.13 (b) but instead show directly that any \( z = \sum_{k=1}^n c_kv_k \in M \) satisfies \( \|x - z\|^2 > \|y - x\|^2 \) unless \( z = y \).

Chapter 7: 2,4,6 and show that \( H^1 \) is complete.
1. Let $A$ be a subset of a metric space $X$.

   a. Give the definition of “$A$ is sequentially compact”.

   b. Give the definition of “$A$ is compact”.

   c. Prove that a compact subset is closed and bounded. You may assume sequential compactness and compactness are equivalent.

   d. Give an example to show that the converse of [c.] is false in general.

   For part d., you could use problem 2a. as follows. Set $A = \{f \in C([0,1]) : \|f\| \leq 1\}$ be the closed unit ball. Then $A$ is closed and bounded. Sequence $\{f_n\}_n$ from problem 2a. is contained in $A$ but it has no uniformly convergent subsequence. For any uniformly convergent subsequence would be pointwise convergent and have to converge to the pointwise limit $f = \chi_{(0,1]}$ of $\{f_n\}_n$, which is not in $C([0,1])$ so, in particular, it is not in $A$.

2. a. Define $f_n : [0,1] \to R$ by $f_n(x) = x^{1/n}$, $n \geq 1$. Does $\{f_n\}_n$ converge uniformly on $[0,1]$? Justify your answer.
\[ f_n \to \chi_{(0,1)} \text{ pointwise but not uniformly since if it did, } \chi_{(0,1)} \text{ would have to be continuous!} \]

b. Show that \( \sum_{n=1}^{\infty} \left( \frac{x}{1+x} \right)^n \sin nx \) is continuous on \([0, \infty)\).

Fix \( L > 0 \) and show that the sum \( g \) belongs to \( C([0, L]) \). Let \( g_n(x) = \left( \frac{x}{1+x} \right)^n \sin nx \). Then \( g_n \in C([0, L]) \) and

\[
\|g_n\|_{\infty} \leq r^n, \quad r = \frac{L}{1+L} < 1
\]

Therefore \( \sum_n \|g_n\|_{\infty} < \infty \) so \( g \in C([0, L]) \) since absolute convergence implies convergence in Banach spaces. As \( L > 0 \) is arbitrary, \( g \) is continuous on \([0, \infty)\).

3.

Consider the space of continuously differentiable functions

\[ C^1([0,1]) = \{ f : [0,1] \to R : f, f' \text{ are continuous} \} \]

with norm

\[ \|f\| = \|f\|_{\infty} + \|f'\|_{\infty} \]

where \( \|g\|_{\infty} = \sup\{|g(x)| : 0 \leq x \leq 1\} \). Show that \( C^1([0,1]) \) is a complete metric space.

4.

a. State the Arzelà-Ascoli Theorem.
b. Let \( F \subset C[0,1] \). Define “\( F \) is (uniformly) equicontinuous family”.

c. Let

\[ F = \{ f \in C[0,1] : f(0) = 0, \quad \frac{|f(x) - f(y)|}{|x - y|^{1/2}} \leq 1, \forall x, y, \ x \neq y \} \]

Prove that \( F \) is compact in \( C[0,1] \).
Material Covered Fall 2008

1. Chapter 1 (fast!)
2. Chapter 2-thru Ascoli-Arzela applied to Peano Thm. for ODEs
3. Chapter 3-added uniform contraction principle, covered Volterra Int. Eqns and BVPs.
4. Chapter 12-added material from Folland to flesh out measure theory; integral convergence theorems, completeness of $L^p$-spaces.
5. Chapter 5-Bounded linear operators-definitions. 5. Chapter 6- Hilbert Spaces
6. Chapter 7-Fourier series