1. Basic Course Information

MAT476 Partial Differential Equations  Spring 2007
Text: Partial Differential Equations of Mathematical Physics
and Integral Equations, Guenther and Lee
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Office Hours: 12:40 Mon. & Wed. or by appt.

Final Exam Wed. May 9, 10-11:50 See Dept. web page for old exams!

The course will cover the standard solution methods for Laplace’s
equation, the heat equation, the wave equation, and their close relatives. These include separation of variables, Fourier series and Fourier
transform methods, and the method of characteristics. Green’s func-
tions, energy methods, and maximum principles will be emphasized.

Homework will be assigned and collected regularly. Midterm and
Final exams will be given. The course grade will be determined by:
1/3 for each exam and 1/3 homework.

You may find materials on Prof. Don Jones web page useful:
http://math.la.asu.edu/~dajones/dajones.html

Note previous final exams for this course posted on:
http://math.asu.edu/grad/diffeqs.html

Other excellent references:

1. Partial Differential Equations, F. John

2. Mathematics Applied to Deterministic Problems in the Natural Sci-
ences, Lin and Segel.

2. Homework assignments

Assignment #1 (Due Friday, Jan 26): Text 1.3: 1,2,4; 1.4: 1;1.8: 3;
exercise 0.3 and exercise 0.4 next pages.

Assignment #2 (Due Friday, Feb 9) Text 2.1: 5; Exercise 0.6 and
exercise 0.5 (see text below).

Also solve each of the following:

\[-xu_x + yu_y = xyu, \quad u(x, 0) = x\]
\[(y + u)u_x + yu_y = x - y, \quad u(x, 1) = 1 + x\]
\[ u_x + u_y + zu_z = u^3, \quad u(x, y, 1) = h(x, y) \]

Assignment #3 (Due Friday, Feb 16) Text 3.1: 1,3,4,7-10,16; 3.2: 1,2,9

Assignment #4 (Due Friday, Feb 23) Text 3.3: 1,2,9

Assignment #5 (Due Friday, Mar 2) Text 4.1: 6,7; 4.2: 1,3,9,10

Also, give a careful argument supporting the displayed estimate on middle pg 91 of text. You will need to use the triangle inequality and the inequality
\[ |\int_a^b g(x)dx| \leq \int_a^b |g(x)|dx. \]

Assignment #6 (Due Friday, Mar 9) Text 4.3: 2,6,7

Assignment #7 (Due Friday, Mar 23) Text 5.1: 1,3; 5.2 1-5.

Assignment #8 (Due Friday, Mar 30) exercise 0.7, Text 5.3: 2, 5.6: 1,2 (read 5.6!)

Assignment #9 (Due Friday, Apr 6) Text 5.1: 5; 8.1 1,10; 8.2 10.

Assignment #10 (Due Friday, Apr 13) Text 8.3: 1,4,11. Also show that the unique solution \( u \) of \( u'' = h(x), \quad 0 \leq x \leq 1 \) satisfying \( u(0) = 0 = u(1) \) can be expressed as \( u(x) = \int_0^1 G(x, y)h(y)dy \) for a Green’s function \( G \). Give an explicit formula for \( G \! \)!

Assignment #11 (Due Friday, Apr. 20) Text 8.3: verify displayed equation for normal derivative of \( G \) above (3-22) page 313; 8.4: 3,4,11. Do exercise 0.2.

Exercise 0.1. Let \( f : D \to \mathbb{R} \) be a real-valued function on an open subset \( D \subset \mathbb{R}^2 \). Suppose that \( f \) has continuous partial derivatives in \( D \) (or more generally, suppose that \( f \) is differentiable in \( D \)) and
\[ \frac{\partial f}{\partial x}(x, y) = 0 = \frac{\partial f}{\partial y}(x, y), \quad (x, y) \in D \]
Prove that \( f \) is constant in \( D \): \( \exists c \in \mathbb{R} \) such \( f(x, y) = c, \quad (x, y) \in D \)? Check your advanced calculus text.

Exercise 0.2. Let \( f : (a, b) \to \mathbb{R} \) be a real-valued function such that \( f'(x) \) exists for \( x \in (a, b), \quad f'(x_0) = 0 \) for some \( x_0 \), and \( f''(x_0) > 0 \). Use the definition of \( f''(x_0) \) as a limit to show that there exists \( \delta > 0 \) such that \( f'(x) > 0 \) for \( x_0 < x < x_0 + \delta \). Then use the Mean Value Theorem to show that \( f(x) > f(x_0) \) for \( x_0 < x < x_0 + \delta \). Use this result to argue that if a differentiable function \( f \) has a local maximum at \( x_0 \) and if \( f''(x_0) \) exists, then \( f''(x_0) \leq 0 \). Use the previous result to argue that if \( u \) is a differentiable function on an open set \( D \subset \mathbb{R}^3 \) which attains a local maximum at \( x_0 \) and if \( \frac{\partial^2 u}{\partial x_j^2}(x_0), \quad j = 1,2,3 \) exist, then \( \Delta u(x_0) \leq 0 \).

Assignment #12 (Due Friday, Apr. 27) Text 9-2:7; 10-2: 8;10-3:1; Also show that eigenvalues of the Laplacian are nonnegative for
3. ADDITIONAL NOTES

3.1. Vector calculus. In vector calculus we learn the identities

\( \nabla \times \nabla \phi = 0 \)
\( \nabla \cdot (\nabla \times E) = 0 \)

where \( \phi : \mathbb{R}^3 \to \mathbb{R} \) is a \( C^2 \) scalar function and \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) is a \( C^2 \) vector-valued function \( E = (E_1, E_2, E_3) \). Recall

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3} \right)
\]

is called the gradient of \( \phi \), \( \nabla \cdot E \) is called the divergence of \( E \) and \( \nabla \times E \) is the curl of \( E \). For divergence and curl,

\[
\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)
\]

is a differential operator that acts just like the corresponding vector product:

\[
(2) \quad \nabla \cdot E = \sum_{i=1}^{3} \frac{\partial E_i}{\partial x_i}
\]
\[
(3) \quad \nabla \times E = \left( \frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}, \frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}, \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \right)
\]

Exercise 0.3. If \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) is \( C^2 \), verify that

\[
\nabla \times (\nabla \times E) = \nabla (\nabla \cdot E) - \nabla^2 E
\]

where \( \nabla^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \), called the Laplacian, operates componentwise.

Recall that the Jacobian of \( E \) is the matrix

\[
J = \begin{pmatrix}
\frac{\partial E_1}{\partial x_1} & \frac{\partial E_1}{\partial x_2} & \frac{\partial E_1}{\partial x_3} \\
\frac{\partial E_2}{\partial x_1} & \frac{\partial E_2}{\partial x_2} & \frac{\partial E_2}{\partial x_3} \\
\frac{\partial E_3}{\partial x_1} & \frac{\partial E_3}{\partial x_2} & \frac{\partial E_3}{\partial x_3}
\end{pmatrix}
\]

From this we see that the divergence of \( E \), \( \nabla \cdot E \) is just the trace of \( J \) (sum of diagonal elements of \( J \)). Any matrix may be written as a sum of a symmetric matrix and an antisymmetric matrix:

\[
J = \frac{1}{2} (J + J^T) + \frac{1}{2} (J - J^T)
\]
Observe that the first summand is symmetric and the second is antisymmetric—its transpose equals its negative. If $\nabla \times E = (a, b, c)$ are the components of the curl of $E$, we see that the non-zero components of the antisymmetric part of $J$ are:

$$\frac{1}{2} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

3.1.1. **Maxwell’s Equations.** Maxwell’s equations for the electric field vector $E$ and magnetic field vector $B$ in free space where charged particles and currents are absent are:

$$\begin{align*}
\nabla \cdot E &= 0 \\
\nabla \times E &= -\frac{\partial B}{\partial t} \\
\nabla \cdot B &= 0 \\
c^2 \nabla \times B &= \frac{\partial E}{\partial t}
\end{align*}$$

(4)

where $c$ is the speed of light.

Taking the curl of both sides of the second equation, applying the identity of exercise 0.3, and using the first then the fourth of Maxwell’s equations, we get

$$-\nabla^2 E = -\nabla \times \left( \frac{\partial B}{\partial t} \right)$$

$$= -\frac{\partial}{\partial t} (\nabla \times B)$$

$$= -c^{-2} \frac{\partial^2 E}{\partial t^2}$$

Therefore, $E = E(x_1, x_2, x_3, t)$ satisfies the wave equation:

$$(5) \quad \frac{\partial^2 E}{\partial t^2} = c^2 \nabla^2 E = c^2 \sum_{i=1}^{3} \frac{\partial^2 E}{\partial x_i^2}$$

Since $E = (E_1, E_2, E_3)$, each component $E_i$ satisfies the wave equation

$$(6) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}$$

A parallel computation shows that $B$ also satisfies (5).
3.1.2. **Conservative vector fields.** We also learn from vector calculus that if $\nabla \times E = 0$ for some $C^1$ vector function, not necessarily the electric field vector, then there is a scalar function $\phi$ such that

$$\nabla \phi = E$$

It is worth noting that $\phi$ must satisfy the system of partial differential equations:

$$\frac{\partial \phi}{\partial x_1} = E_1, \quad \frac{\partial \phi}{\partial x_2} = E_2, \quad \frac{\partial \phi}{\partial x_3} = E_3$$

We solve these equations using Stokes’ Theorem

$$\oint_{\partial S} E \cdot dr = \iint_S (\nabla \times E) \cdot ndS$$

where $S$ is a surface in $\mathbb{R}^3$, oriented by the unit normal vector $n$, with smooth boundary $\partial S$ oriented by the right-hand rule compatible with $n$ (see your vector calculus text). Because $\nabla \times E = 0$, Stokes’ Theorem says that the line integral of $E$ around every closed curve is zero and therefore $E$ is conservative, meaning that the line integral

$$\int_q^p E \cdot dr$$

along any path from point $q$ to point $p$ is the same. It is easy to see that if $p = (x_1, x_2, x_3)$ and $q$ is a fixed point (e.g., $q = (0, 0, 0)$), then

$$\phi(p) = \int_q^p E \cdot dr$$

satisfies $\nabla \phi = E$ and $\phi(q) = 0$. If $E$ were defined on a sub-domain $\Omega$ of $\mathbb{R}^3$, then we can draw the same conclusion provided that $\Omega$ is simply connected: every closed curve in $\Omega$ is homotopic in $\Omega$ to a single point.

**Exercise 0.4.** Show that $\phi$, defined by the integral above, satisfies the system (7). Hint: check out your vector calculus or physics text.

Thus we see that for a $C^1$ vector function $E$ in $\mathbb{R}^3$, $E = \nabla \phi$ for some $C^2$ scalar function $\phi$ if and only if $\nabla \times E = 0$ in $\mathbb{R}^3$.

In electrostatics, one assumes that the electric and magnetic field vectors do not depend on time. In that case, the second of (4) gives $\nabla \times E = 0$ so $E$ is conservative. It follows that $E$ is the gradient of some function $\phi$, called the electrostatic potential:

$$E = \nabla \phi$$
By the first of equations (4), we see that \( \phi \) satisfies Laplace’s equation
\[
0 = \nabla^2 \phi = \sum_{i=1}^{3} \frac{\partial^2 \phi}{\partial x_i^2}
\]

3.1.3. Divergence-free vector fields. What can we say about a \( C^1 \) vector function \( E \) on \( \mathbb{R}^3 \) if we know that
\[
\nabla \cdot E = 0.
\]
From Maxwell’s equations we see that both the electric field and the magnetic field vector are divergence free.

By the second identity of (1), we know that the divergence of a curl is zero. Could it be true that since the divergence of \( E \) is zero, then \( E \) must be a curl of some other vector function? You will prove the following result

**Proposition 1.** Let \( E : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a \( C^1 \) vector-valued function satisfying
\[
\nabla \cdot E = 0
\]
in \( \mathbb{R}^3 \). Then there exists a \( C^1 \) vector-valued function \( A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that
\[
E = \nabla \times A
\]
A can be viewed as a vector potential function for \( E \).

Notice that \( A = (A_1, A_2, A_3) \) must satisfy the system of partial differential equations
\[
\begin{align*}
\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} &= E_1 \\
\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} &= E_2 \\
\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} &= E_3
\end{align*}
\]
(9)

The following result shows that (8) does not have a unique solution.

**Lemma 1.** If (8) has a solution \( A(x_1, x_2, x_3) \) and if \( \phi \) is any \( C^2 \) scalar function, then \( A + \nabla \phi \) is also a solution.

If (8) has two solutions \( A(x_1, x_2, x_3) \) and \( B(x_1, x_2, x_3) \), then there is a scalar function \( \phi \) such that
\[
A - B = \nabla \phi
\]

**Proof.** The first assertion follows from (1). The second assertion follows immediately from the fact that \( \nabla \times (A - B) = 0 \) and our vector calculus review above. \( \square \)
Corollary 1. If (8) has a $C^2$ solution $A = (A_1, A_2, A_3)$, then it has a $C^1$ solution $B$ with $B_1 \equiv 0$ in $\mathbb{R}^3$.

Proof. By Lemma, $B = A - \nabla \phi$ is a solution for any $C^2$ scalar function $\phi$. We can choose a $C^2$ scalar function $\phi$ such that
\[
\frac{\partial \phi}{\partial x_1} = A_1(x_1, x_2, x_3)
\]
Clearly, then $B_1 = 0$. □

In view of the Corollary, we may as well try to solve (8) for $A = (0, A_2, A_3)$. We need only find two functions rather than three! The equations for $A_2, A_3$ from (9) are:

\[
\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = E_1
\]
\[
\frac{\partial A_3}{\partial x_1} = -E_2
\]
\[
\frac{\partial A_2}{\partial x_1} = E_3
\]

Exercise 0.5. Solve the system of equations (10). Hint: first solve the last two equations to find $A_2$ and $A_3$ and then show that the first equation automatically holds—well, almost!
3.2. Flow of Ideal Gases. This brief section is devoted to fleshing out Chapter 1, section 7 of our text. It presumes a substantial knowledge of the theory of ordinary differential equations. I suggest not to read it!

We follow the time evolution of a bounded volume of gas particles with density $\rho(x, t)$ as it moves under the velocity field $v(x, t)$. Recall that conservation of mass gives

$$\rho_t + \nabla \cdot (\rho v) = 0$$

If $V_{t_0}$ denotes this volume at $t = t_0$, let $V_t$ be the corresponding volume of particles at time $t$. In order to express $V_t$, we need the particle trajectory $x(t, t_0, x_0)$ which passes through $x_0$ at time $t_0$. It satisfies the ODE

$$x' = v(x, t), \quad x(t_0) = x_0$$

Then

$$V_t = x(t, t_0, \bullet)(V_{t_0}) = \{x(t, t_0, x_0): x_0 \in V_{t_0}\}$$

It will be useful to express integrals such as $\int_{V_t} f(y)dy$ as integrals over the original domain $V_{t_0}$ by the change of variables $y = g(x_0) \equiv x(t, t_0, x_0)$. Recall this formula is given by

$$\int_{V_t} f(y)dy = \int_{g(V_{t_0})} f(y)dy = \int_{V_{t_0}} (f \circ g)(x_0)| \det Jg(x_0)|dx_0$$

where $Jg$ is the Jacobian matrix of $g$. Consequently, we need an expression for $Jg$. This comes from basic ODE theory since

$$Jg = X(t) \equiv \frac{\partial x(t, t_0, x_0)}{\partial x_0}$$

satisfies the matrix differential equation

$$X' = A(t)X, \quad X(t_0) = I$$

where

$$A(t) = \frac{\partial v(x(t, t_0, x_0), t)}{\partial x}$$

By Abel’s Theorem

$$\det Jg = \det X(t) = \det X(t_0) \exp \left(\int_{t_0}^t \sum a_{ii}(s)ds\right)$$

$$= \exp \left(\int_{t_0}^t (\nabla_x \cdot v)(x(s, t_0, x_0), s)ds\right)$$
3. ADDITIONAL NOTES

Suppose \( f(y, t) \) is a vector function. Then

\[
\int_{V_i} f(y, t) dy = \int_{V_{i_0}} f(x(t, t_0, x_0), t) \exp \left( \int_{t_0}^{t} (\nabla_x \cdot v)(x(s, t_0, x_0), s) ds \right) dx_0
\]

Differentiating at \( t = t_0 \) we get

\[
\frac{d}{dt} \bigg|_{t=t_0} \int_{V_i} f(y, t) dy = \int_{V_{i_0}} [\nabla \cdot (f v) + f_t + (\nabla_x \cdot v)f] dx_0
\]

\[
= \int_{V_{i_0}} \nabla \cdot (fv) + f_t dx_0
\]

\[
= \int_{V_{i_0}} f_t dx_0 + \int_{\partial V_{i_0}} f v \cdot ndS
\]

We do not need the last expression as the middle one is great.

Now we can express the \( i \)-th component of the momentum of this collection of particles as

\[
M_i(t) = \int_{V_i} (\rho v_i)(y, t) dy
\]

Differentiating with respect to \( t \), evaluating at \( t = t_0 \), and using (11) gives

\[
\dot{M}_i(t_0) = \int_{V_{i_0}} \nabla \cdot (\rho v_i v) + (\rho v_i)_t dx_0
\]

\[
= \int_{V_{i_0}} \nabla \cdot (\rho v_i v) + \rho v_i + \rho v_{it} dx_0
\]

\[
= \int_{V_{i_0}} \nabla \cdot (\rho v_i v) - (\nabla \cdot \rho v)v_i + \rho v_{it} dx_0
\]

\[
= \int_{V_{i_0}} \rho [(v \cdot \nabla) v_i + v_{it}] dx_0
\]

This must equal the \( i \)-th component of the forces acting on \( V_{i_0} \) which are given by

\[
-\int_{V_{i_0}} \rho dx_0 g - \int_{\partial V_{i_0}} pmdS = -\int_{V_{i_0}} \rho dx_0 g - \int_{V_{i_0}} \nabla p dx_0
\]

See our text, \( p \) denotes pressure, \( g \) denotes gravitational constant vector. Since \( V_{i_0} \) was arbitrary, we get the partial differential equation:

\[
\rho [v_i + (v \cdot \nabla)v] = -\nabla p - \rho g
\]
3.3. Equations of Demography. Let \( u(a, t) \) be the age-density of a (human or animal) population. Here \( a \geq 0 \) denotes age and \( t \geq 0 \) denotes time. By this we mean that, given an age range \( a_1 < a < a_2 \) and time \( t \) then

\[
\int_{a_1}^{a_2} u(a, t) da = \text{number of individuals with ages } a_1 < a < a_2 \text{ at time } t
\]

The total size of the population at time \( t \) would then be

\[
\int_0^\infty u(a, t) da
\]

If you wanted to know the size of the cohort of individuals which at time \( t_0 \) have ages between \( a_1 \) and \( a_2 \) as they age, i.e., for \( t > t_0 \), you would write:

\[
\int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} u(a, t) da
\]

Notice that when \( t = t_0 \) in the above expression, we get the size of this cohort at time \( t_0 \). Differentiating this expression, using Leibniz formula and the fundamental theorem of calculus, we get

\[
\frac{d}{dt} \int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} u(a, t) da = \int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} u_t(a, t) da + u(a_2 + (t-t_0), t) - u(a_1 + (t-t_0), t)
\]

\[
= \int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} [u_t(a, t) + u_a(a, t) - \mu(a, t)] da
\]

The only change can be caused by death since no one is born into this cohort. If

\[
\mu(a, t) = \text{per capita death rate at time } t \text{ of individuals of age } a
\]

then

\[
\mu(a, t) u(a, t) da = \text{death rate at time } t \text{ of individuals of age } a
\]

and the time derivative computed above must be

\[
\frac{d}{dt} \int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} u(a, t) da = - \int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} \mu(a, t) u(a, t) da
\]

Consequently, we find that

\[
\int_{a_1 + (t-t_0)}^{a_2 + (t-t_0)} u_t(a, t) + u_a(a, t) + \mu(a, t) u(a, t) da = 0
\]
Evaluating this at $t = t_0$, we get
$$\int_{a_1}^{a_2} u_t(a, t_0) + u_a(a, t_0) + \mu(a, t_0) u(a, t_0) da = 0$$
and since $a_1, a_2$ and $t_0$ were arbitrary, $u$ must satisfy
$$u_t(a, t) + u_a(a, t) + \mu(a, t) u(a, t) = 0, \quad 0 < a < \infty, \quad t > 0$$

It seems reasonable that we should be given the age-density of the population at time $t = 0$:
$$u(a, 0) = u_0(a), \quad a > 0$$
In addition, we must account for births of individuals of age $a = 0$. Suppose that we are given the number of births, $B(t)$, at time $t$. Then we must have
$$u(0, t) = B(t), \quad t > 0$$
We are lead to hope that the following is a well-posed problem for $u$
$$(12) \quad u_t(a, t) + u_a(a, t) + \mu(a, t) u(a, t) = 0, \quad 0 < a < \infty, \quad t > 0$$
$$u(a, 0) = u_0(a), \quad a > 0$$
$$u(0, t) = B(t), \quad t > 0$$

**Exercise 0.6.** Solve (12) when $u_0(a) = 10e^{-a}$, $\mu(a, t) = a$ and $B(t) = 1$. Is your solution continuous? If not, where is it not continuous?

Equations (12) are unsatisfactory because we cannot predict births—they are simply assumed given. A better alternative is to prescribe a birth rate:
$$\beta(a, t) = \text{per capita birth rate at time } t \text{ of individuals of age } a$$
Then we have a formula for $B(t)$:
$$B(t) = \int_0^\infty \beta(a, t) u(a, t) da$$
Putting this into (12) yields a more challenging problem!
3.4. Second Order Constant Coefficients. Consider the linear second order differential equation

\[ au_{xx} + 2bu_{xy} + cu_{yy} = hu + du_x + eu_y + f(x, y) \]

for the function \( u = u(x, y) \). Here, the coefficients \( a, b, c, d, e, h \) are real constants, with \( a, b, c \) not all zero.

Following the same reasoning used to show that the quadratic form

\[ ax^2 + 2bxy + cy^2 = f + dx + ey \]

represents an ellipse, a hyperbola, or parabola, we will obtain standard forms for (13).

Let \( A \) be the symmetric matrix defined by

\[ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \]

Then the left side of (14) can be expressed as \( A z \cdot z \) where \( z = (x, y)^T \) and superscript \( T \) denotes transpose. If \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \), note that \( a = Ae_1 \cdot e_1 \), \( b = Ae_1 \cdot e_2 \), \( c = Ae_2 \cdot e_2 \).

For first order equations, we have to give the value of \( u \) along certain curves and then the PDE will allow us to calculate \( u \) nearby. For second order equations we should expect to have to give \( u \) and perhaps one more piece of info along a curve.

Suppose we give the value of \( u \) along the \( x \)-axis: \( u(x, 0) = g(x) \). From this we may immediately compute \( u_x(x, 0) = g'(x) \) so we get one derivative for free. Does the equation (13) together with the auxiliary condition \( u(x, 0) = g(x) \) allow us to compute all partial derivatives of \( u \), say at \((0, 0)\)? If so, then we might expect to compute \( u \) near \((0, 0)\) from its Taylor series expansion. It seems clear that we have no way of computing \( u_y(x, 0) \) from anything that we are given. Hence we are lead to conclude that we must specify both \( u \) and the value of its derivative normal to the curve, namely \( u_y(x, 0) \), along the \( x \)-axis:

\[ u(x, 0) = g(x), \quad u_y(x, 0) = k(x) \]

where \( g, k \) are given functions. From this information, we may compute the derivatives:

\[ u_x(x, 0) = g'(x), \quad u_{xx}(x, 0) = g''(x), \quad u_{yx}(x, 0) = k'(x) \]

But we need help from (13) to compute \( u_{yy}(x, 0) \)! It gives

\[ cu_{yy}(x, 0) = hg(x) + dg'(x) + ek(x) + f(x, 0) - 2bk'(x) - ag''(x) \]

Thus, if \( c = Ae_2 \cdot e_2 \neq 0 \) we may compute all derivatives of \( u \) along the \( x \)-axis up to the second order. It is not so hard to show that in fact we
may compute all partial derivatives of \( u \) if \( c \neq 0 \). Therefore, \( u \) should be completely determined, at least near the \( x \)-axis, if \( c \neq 0 \).

We are in big trouble though if \( c = A e_2 \cdot e_2 = 0! \) This suggests that vectors \( v \) satisfying \( A v \cdot v = 0 \) may be in some way special for (13). We come back to this point later.

It is standard practice to seek new coordinates that will simplify a differential equation. Lets try an orthogonal transformation given by

\[
(x, y)^T = V (r, s)^T
\]

where

\[
V = \{v_{ij}\}
\]

is an orthogonal matrix whose columns \( v_1 = (v_{11}, v_{21})^T \) and \( v_2 = (v_{12}, v_{22})^T \) give an orthonormal basis. Then \( V^{-1} = V^T \).

We mat write the change of variables more simply as:

\[
x = rv_{11} + sv_{12}\\
y = rv_{21} + sv_{22}
\]

Its inverse is given by

\[
(r, s)^T = V^T (x, y)^T
\]
or,

\[
r = xv_{11} + yv_{21}\\
s = xv_{12} + yv_{22}
\]

Let

\[
U(r, s) = u(rv_{11} + sv_{12}, rv_{21} + sv_{22}) = (u \circ V)(r, s)
\]

where now we abuse notation by regarding \( V \) as a mapping \( V : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by the rule \((r, s) \rightarrow (x, y) = (rv_{11} + sv_{12}, rv_{21} + sv_{22})\). Then

\[
u(x, y) = U(xv_{11} + yv_{21}, xv_{12} + yv_{22}) = (U \circ V^T)(x, y)
\]

with the same abuse of notation. By the chain rule:

\[
Du = DU \circ DV^T
\]
or, more simply,

\[
\begin{align*}
  u_x &= U_r r_x + U_s s_x = U_r v_{11} + U_s v_{12} \\
  u_y &= U_r r_y + U_s s_y = U_r v_{21} + U_s v_{22}
\end{align*}
\]

from which we compute:

\[
\begin{align*}
  u_{xx} &= [U_{rr} r_x + U_{rs} s_x] v_{11} + [U_{sr} r_x + U_{ss} s_x] v_{12} \\
         &= U_{rr} v_{11}^2 + 2U_{rs} v_{11} v_{12} + U_{ss} v_{12}^2
\end{align*}
\]
and
\[ u_{xy} = [U_{rr}r_y + U_{rs}r_y]v_{11} + [U_{sr}r_y + U_{ss}r_y]v_{12} = U_{rr}v_{11}v_{21} + U_{rs}[v_{11}v_{22} + v_{12}v_{22}] + U_{ss}v_{12}v_{22} \]

and
\[ u_{yy} = [U_{rr}r_y + U_{rs}r_y]v_{21} + [U_{sr}r_y + U_{ss}r_y]v_{22} = U_{rr}v_{21}^2 + 2U_{rs}v_{21}v_{22} + U_{ss}v_{22}^2 \]

Hence
\[ au_{xx} + 2bu_{xy} + cu_{yy} = a[U_{rr}v_{11}^2 + 2U_{rs}v_{11}v_{12} + U_{ss}v_{12}^2] + 2b[U_{rr}v_{11}v_{21} + U_{rs}[v_{11}v_{22} + v_{12}v_{22}] + U_{ss}v_{12}v_{22}] + c[U_{rr}v_{21}^2 + 2U_{rs}v_{21}v_{22} + U_{ss}v_{22}^2] = U_{rr}[av_{11}^2 + 2bv_{11}v_{21} + cv_{21}^2] + U_{rs}[2av_{11}v_{12} + 2b(v_{11}v_{22} + v_{12}v_{22}) + 2cv_{21}v_{22}] + U_{ss}[av_{12}^2 + 2bv_{12}v_{22} + cv_{22}^2] = U_{rr}[Av_1 \cdot v_2] + U_{rs}[Av_1 \cdot v_2] + U_{ss}[Av_2 \cdot v_2] \]

We have considerable freedom in choosing the orthogonal matrix \( V \). However, one choice stands out. Let \( v_i, \ i = 1, 2 \) be unit-length eigenvectors for \( A \) corresponding to eigenvalues \( \lambda_i \).
\[ Av_i = \lambda_i v_i, \ i = 1, 2 \]

Since \( A \) is symmetric, \( v_1 \cdot v_2 = 0 \). They give an orthonormal basis for \( \mathbb{R}^2 \) and \( V \) diagonalizes \( A \)
\[ AV = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \]

This \( V \) leads to
\[ Av_1 \cdot v_1 = \lambda_1, \ Av_1 \cdot v_2 = 0, \ Av_2 \cdot v_2 = \lambda_2 \]

Therefore,
\[ au_{xx} + 2bu_{xy} + cu_{yy} = \lambda_1 U_{rr} + \lambda_2 U_{ss} \]

In the new coordinates
\[ f(x, y) = f(rv_{11} + sv_{12}, rv_{21} + sv_{22}) \equiv F(r, s) \]
\[ du_x + eu_y + hu = d(U_rv_{11} + U_sv_{12}) + e(U_rv_{21} + U_sv_{22}) + hU \]
\[ = \hat{d}U_r + \hat{e}U_s + hU \]

Hence, in the new coordinates, (13) becomes:
\[ \lambda_1 U_{rr} + \lambda_2 U_{ss} = F(r, s) + \hat{d}U_r + \hat{e}U_s + hU \]

This does not seem impressive for all the work but let’s continue.
3. ADDITIONAL NOTES

There are three cases:

(a) Elliptic case: \( \lambda_1 \lambda_2 > 0 \).

(b) Hyperbolic case: \( \lambda_1 \lambda_2 < 0 \).

(c) Parabolic case: \( \lambda_1 = 0 \neq \lambda_2 \)

In any case, we may assume, after possibly multiplying (13) by \(-1\) so the signs of \(a, b, c\) are reversed, that \(\lambda_2 > 0\), \(i = 1, 2\).

In the **elliptic case**, where both eigenvalues are positive, we make one further change of variables:

\[
r = \sqrt{\lambda_1} X, \quad s = \sqrt{\lambda_2} Y
\]

and

\[
w(X, Y) = U(r, s)
\]

The resulting equation for \(w\) is:

\[
(18) \quad w_{XX} + w_{YY} = G(X, Y) + \bar{d}w_X + \bar{e}w_Y + hw
\]

where \(G(X, Y) = F(\sqrt{\lambda_1} X, \sqrt{\lambda_2} Y)\). If the entire right-hand side vanishes, we get Laplace’s equation:

\[
w_{XX} + w_{YY} = 0
\]

If all but \(G\) vanishes on the right side, we get Poisson’s equation:

\[
w_{XX} + w_{YY} = G
\]

In the **hyperbolic case**, \(\lambda_1 < 0 < \lambda_2\), we change variables:

\[
r = \sqrt{|\lambda_1|} X, \quad s = \sqrt{\lambda_2} Y
\]

and

\[
w(X, Y) = U(r, s)
\]

The resulting equation for \(w\) is:

\[
(19) \quad -w_{XX} + w_{YY} = G(X, Y) + \bar{d}w_X + \bar{e}w_Y + hw
\]

where, hopefully, the reader can express \(G\) in terms of \(F\) as we did in the elliptic case. If the entire right side vanishes, this is the wave equation:

\[
-w_{XX} + w_{YY} = 0
\]

We could change variables again by

\[
\xi = X - Y, \quad \eta = X + Y
\]

and

\[
W(\xi, \eta) = w(X, Y)
\]

to obtain another standard form of the wave equation in two variables:

\[
(20) \quad W_{\xi\eta} = H(\xi, \eta) + \bar{d}W_X + \bar{e}W_Y + hW
\]
In the parabolic case, the equation for $U(r, s)$ becomes

\begin{equation}
U_{ss} = \lambda_2^{-1}[F(r, s) + \hat{d}U_r + \hat{e}U_s + hU]
\end{equation}

3.5. Energy Method for Wave Equation. We show how the energy method can be used to establish uniqueness of solutions of the wave equation:

\begin{align*}
0 &= u_{tt} - c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\
u(x, 0) &= f(x), \quad -\infty < x < \infty \\
u_t(x, 0) &= g(x), \quad -\infty < x < \infty
\end{align*}

By superposition, it suffices to show that $u = 0$ is the only solution when $f = g = 0$.

Fix a point $P = (x', t')$ with $t' > 0$. The characteristics through $P$ take the form

$$x - ct = x' - ct'$$

$$x + ct = x' + ct'$$

Together with the interval $[x' - ct', x' + ct']$ on the $x$-axis in the $(x, t)$-plane, the characteristics bound an isosceles triangle $T$ with $P$ at its apex.
Multiplying the wave equation by \( u_t \) and integrating over \( T \) we have

\[
0 = \int_T \int u_t u_{tt} - c^2 u_t u_{xx} \, dx \, dt
\]

\[
= \int_0^{\prime} \left( \int_{x' - ct'}^{x' + ct'} u_t u_{tt} \, dx - c^2 \int_{x' - ct'}^{x' + ct'} u_t u_{xx} \, dx \right) \, dt
\]

\[
= \int_0^{\prime} \int_{x' - ct'}^{x' + ct'} \left[ \frac{u_t^2}{2} + c^2 \frac{u_x^2}{2} \right] \, dxdt
\]

\[
-c^2 \int_0^{\prime} [(u_t u_x)(x' + ct' - ct, t) - (u_t u_x)(x' - ct' + ct, t)] dt
\]

\[
= \frac{1}{2} \int_{x' - ct'}^{x'} [u_t^2 + c^2 u_x^2] (x, \frac{x - x' + ct'}{c}) \, dx + \frac{1}{2} \int_{x' - ct'}^{x'} [u_t^2 + c^2 u_x^2] (x, \frac{x' - x + ct'}{c}) \, dx
\]

\[
-c^2 \int_0^{\prime} [(u_t u_x)(x' + ct' - ct, t) - (u_t u_x)(x' - ct' + ct, t)] dt
\]

\[
= \frac{c}{2} \int_0^{\prime} [u_t^2 + c^2 u_x^2] (cs + x' - ct', s) ds + \frac{c}{2} \int_0^{\prime} [u_t^2 + c^2 u_x^2] (x' + ct' - cs, s) ds
\]

\[
-c^2 \int_0^{\prime} [(u_t u_x)(x' + ct' - ct, t) - (u_t u_x)(x' - ct' + ct, t)] dt
\]

\[
= \frac{1}{2} \int_0^{\prime} \left[ c(u_t^2 + c^2 u_x^2) + 2c^2 u_t u_x \right]_{(x' + ct - ct', t)} dt
\]

\[
+ \frac{1}{2} \int_0^{\prime} \left[ c(u_t^2 + c^2 u_x^2) - 2c^2 u_t u_x \right]_{(x' + ct' - ct, t)} dt
\]

\[
= \frac{c}{2} \int_0^{\prime} (u_t + cu_x)^2 (x' + ct - ct', t) dt + \frac{c}{2} \int_0^{\prime} (u_t - cu_x)^2 (x' + ct' - ct, t) dt
\]

Integration by parts is followed by a change of order of integration and finally by a change of variable to obtain the final expression. It follows that

\[
0 = u_t + cu_x, \quad (x, t) = (x' + ct - ct', t), \quad 0 \leq t \leq t'
\]

\[
0 = u_t - cu_x, \quad (x, t) = (x' + ct' - ct, t), \quad 0 \leq t \leq t'
\]

In particular,

\[
u_t(x', t') = u_x(x', t') = 0
\]

and as \( P \) was arbitrarily chosen, we find that these derivatives vanish for all \((x', t')\) with \( t' > 0 \). It follows that \( u \) is a constant and identically zero by the initial conditions.

\[ F(x,t) = u_{tt} - c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \]

\[ u(x,0) = 0, \quad -\infty < x < \infty \]

\[ u_t(x,0) = 0, \quad -\infty < x < \infty \]

\[ F(x,t) = v_t - cv_x \]

\[ v(x,t) = u_t + cu_x \]

\[ u_t(x,0) = 0 = v(x,0), \quad -\infty < x < \infty \]

Method of characteristics gives

\[ v(x,t) = \int_0^t F(x+ct-c\eta, \eta) d\eta \]

and

\[ u(x,t) = \int_0^t \int_0^\xi F(x-ct+2c\xi-c\eta, \eta) d\eta d\xi \]

\[ = \int \int_{RT} F \circ G(\xi, \eta) d\eta d\xi \]

\[ = \int \int_{G(RT)} F \circ G \circ G^{-1}(x', t') | \det DG^{-1} | dx' dt' \]

\[ = \frac{1}{2c} \int \int_T F(x', t') dx' dt' \]

where

\[(x', t')^* = G(\xi, \eta) = (x-ct+2c\xi-c\eta, \eta)^* = \left( \begin{array}{cc} 2c & -c \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \xi \\ \eta \end{array} \right) + \left( \begin{array}{c} x-ct \\ 0 \end{array} \right) \]

maps the right triangle \( RT = \{(\xi, \eta) : 0 \leq \eta \leq \xi \leq t\} \) onto the isosceles triangle \( T(x, t) \) formed by the characteristic curves \( x' - ct' = x - ct \) and \( x' + ct' = x + ct \) and the segment \( t' = 0 \) and \( x - ct \leq x' \leq x + ct \) in the \( x' - t' \)-plane. So

(22)

\[ u(x,t) = \int_0^t \int_{x-c(t-t')}^{x+c(t-t')} F(x', t') dx' dt' \]

As an example, suppose that some nonzero forcing occurs near \( x = x_0 \) at time near \( t = t_0 > 0 \), e.g.:

\[ F(x,t) = \begin{cases} \frac{1}{\pi r^2}, & (x-x_0)^2 + (t-t_0)^2 < r^2 \\ 0, & (x-x_0)^2 + (t-t_0)^2 > r^2 \end{cases} \]

where \( r \) is a very small constant: \( r << t_0 \). Suppose that you are at position \( x' > x_0 \). When will you know about this event? Equation (22)
implies that $u = 0$ until such time as the triangle $T(x', t)$ includes a point of the disk $(x - x_0)^2 + (t - t_0)^2 < r^2$. Hence, for very small $r > 0$

$$u(x, t) \approx \begin{cases} 0, & t < t_0 + \frac{x' - x_0}{c} \\ 1, & t > t_0 + \frac{x' - x_0}{c} \end{cases}$$

This seems strange! The wave equation describes sound waves ($c = 1100$ ft/sec). We imagine the solution above as the result of a firecracker burst at $x = x_0$ when $t = t_0$. Our solution shows that we do not hear anything for a while and then we hear forever after, and with no diminution, the result of the explosion!

### 3.7. Midterm Exam Solutions.

1. Find the solution, if it exists, of

$$u_x + u_t = x + t^2, \quad u(x, 0) = \frac{1}{1 + x^2}$$

Characteristics:

$$x = \tau + s, \quad t = \tau$$

so

$$\frac{du}{d\tau} = \tau + s + \tau^2$$

thus

$$u = \frac{\tau^2}{2} + s\tau + \frac{\tau^3}{3} + \frac{1}{1 + s^2} = \frac{t^2}{2} + t(x - t) + \frac{t^3}{3} + \frac{1}{1 + (x - t)^2}$$

2. Consider the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty$$

$$u_t(x, 0) = g(x), \quad -\infty < x < \infty$$

a. State d’Alembert’s solution.

b. Let $u_1(x, t)$ be a solution of the initial value problem corresponding to initial data $f_1(x)$ and $g_1(x)$ as above and let $u_2(x, t)$ be a solution of the initial value problem corresponding to initial data $f_2(x)$ and $g_2(x)$. Suppose that

$$|f_1(x) - f_2(x)| \leq \epsilon, \quad |g_1(x) - g_2(x)| \leq \epsilon, \quad -\infty < x < \infty$$

Show that

$$|u_1(x, t) - u_2(x, t)| \leq (1 + T)\epsilon, \quad 0 \leq t \leq T, \quad -\infty < x < \infty$$
c. Let
\[ f(x) = \begin{cases} 
\cos(x), & x \in [-\pi/2, \pi/2] \\
0, & x \notin [-\pi/2, \pi/2]
\end{cases} \]
and suppose \( g(x) = 0 \) for all \( x \). Find all \( t \) such that \( u(8\pi, t) > 0 \). Justify your answer!

Since
\[ u(8\pi, t) = \frac{f(8\pi + ct) + f(8\pi - ct)}{2} \]
we need to consider each summand:
\[
f(8\pi + ct) > 0
\]
\[ \iff -\pi/2 < 8\pi + ct < \pi/2 \]
\[ \iff -\frac{17\pi}{2} < ct < -\frac{15\pi}{2} \]
\[ \iff -\frac{17\pi}{2c} < t < -\frac{15\pi}{2c} \]
and
\[
f(8\pi - ct) > 0
\]
\[ \iff -\pi/2 < 8\pi - ct < \pi/2 \]
\[ \iff -\frac{17\pi}{2} < -ct < -\frac{15\pi}{2} \]
\[ \iff -\frac{17\pi}{2c} < -t < -\frac{15\pi}{2c} \]
\[ \iff \frac{15\pi}{2c} < t < \frac{17\pi}{2c} \]
Note that both summands are positive or zero so there are no cancellations. So
\[ u(8\pi, t) > 0 \iff -\frac{17\pi}{2c} < t < -\frac{15\pi}{2c} \text{ or } \frac{15\pi}{2c} < t < \frac{17\pi}{2c} \]

3. Let \( f(x) \) be a \( 2L \)-periodic function on \( -\infty < x < \infty \).

a. Using the relations
\[
0 = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx, \ n \neq m
\]
\[
0 = \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx, \ \text{all } m, n
\]
\[
L = \int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right) dx = \int_{-L}^{L} \cos^2\left(\frac{m\pi x}{L}\right) dx, \ m \geq 1.
\]
3. ADDITIONAL NOTES

show how we formally obtain the coefficients \(a_n, b_n\) in the Fourier series

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)
\]

b. What does it mean that the Fourier series (23) converges to \(f(x)\) uniformly on the interval \([a, b]\)? Give a mathematically precise definition!

If

\[
S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)
\]

denotes the finite sum of the first \(m + 1\) terms (including \(a_0\)) of (23), then the Fourier series converges uniformly on \([a, b]\) means that for every \(\epsilon > 0\) there exists a positive integer \(N\) such that

\[
|f(x) - S_m(x)| < \epsilon, \quad m \geq N, \quad a \leq x \leq b.
\]

4. Consider the ODD 4-periodic extension of the function

\[
f(x) = \begin{cases} 
1, & 0 \leq x \leq 1 \\
2 - x, & 1 < x \leq 2
\end{cases}
\]

a. Give its Fourier series and the formula for all coefficients. Evaluate only those coefficients that are zero due to even-ness or odd-ness considerations.

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)
\]

where

\[
b_n = \int_{0}^{2} f(x) \sin\left(\frac{n\pi x}{2}\right)dx
\]

b. Give the sum of the series for each \(x \in [0, 2]\), carefully justifying your answers. Be sure to explicitly verify ALL hypotheses of any theorems you use. Where does the series converge uniformly? Justify your answer as above.
After CAREFULLY checking that \( f \) and \( f' \) are piecewise continuous, in particular noting the values of 
\[ f(0^-), f(0^+), f(2^-), f(2^+), f'(0^-), f'(0^+), f'(1^-), f'(1^+), f'(2^-), f'(2^+) \]
and verifying they are finite, we can apply Dirichlet’s Theorem. The Fourier series converges to:
\[
\frac{f(0^-) + f(0^+)}{2} = 0 \text{ at } x = 0
\]
and to \( f(x) \) at each \( x \in (0, 2] \). It converges uniformly to \( f \) on any interval \([a, b]\) where \( 0 < a < b < 4 \) and all translates by \( 4z \) for integer \( z \).

3.8. On Term-by-Term Differentiation of Series. The separation of variables technique applied to linear second order pdes ultimately leads to formal series “solutions” whose validity must be verified. For this our text supplies Proposition 2-3, page 59:

**Proposition 2.** Let \( g_n(x) \) be continuously differentiable on an interval \( I \) and assume the following:

(i) \( \sum_{n=0}^{\infty} g_n(a) \) converges at some point \( a \in I \);
(ii) \( \sum_{n=0}^{\infty} g'_n(x) \) is uniformly convergent on \( I \).

Then \( \sum_{n=0}^{\infty} g_n(x) \) is differentiable on \( I \) and

\[
\left( \sum_{n=0}^{\infty} g_n(x) \right)' = \sum_{n=0}^{\infty} g'_n(x)
\]

As we will see, it is useful to know that the hypotheses of the Proposition imply the additional fact that the series

\[
\sum_{n=0}^{\infty} g_n(x)
\]

converges uniformly on bounded subintervals of \( I \). Indeed, if

\[
S_n(x) = \sum_{k=0}^{n} g_k(x)
\]

is the \( n \)-th partial sum of the series, we have

\[
S_n(x) - S_m(x) = ([S_n(x) - S_m(x)] - [S_n(a) - S_m(a)]) + [S_n(a) - S_m(a)]
\]

\[
= [S'_n(c) - S'_m(c)](x - a) + [S_n(a) - S_m(a)]
\]

for some \( c \) between \( a \) and \( x \) by the Mean Value Theorem. Let \( x \) and \( a \) belong to a bounded subinterval \( J \) of \( I \) of length \( L \). By the uniform convergence of \( \sum_{n=0}^{\infty} g'_n(x) \), if \( \epsilon > 0 \) then there exists a natural number
\[ N \text{ such that } |S'_n(x) - S'_m(x)| < \epsilon/2L \text{ for all } x \in J, \ n \geq N. \] We can also assume \( N \) is so large that \( |S_n(a) - S_m(a)| < \epsilon/2 \) for \( n \geq N \). Taking absolute values in the inequality above, using the triangle inequality and our estimates results in \( |S_n(x) - S_m(x)| < \epsilon \) for \( n \geq N \), proving the uniform convergence of \( \sum_{n=0}^{\infty} g_n(x) \).

Our differential equations are typically of second order so we are often faced with showing that \( \sum_{n=0}^{\infty} g_n(x) \) is twice differentiable and each derivative is given by term-by-term differentiation. The following result is stated so as to be maximally useful in this context.

**Proposition 3.** Let \( g_n(x) \) be twice continuously differentiable on an interval \( I \) and assume the following:

(i) \( \sum_{n=0}^{\infty} g_n(a) \) converges at some point \( a \in I \);
(ii) \( \sum_{n=0}^{\infty} g'_n(b) \) converges at some point \( b \in I \);
(iii) \( \sum_{n=0}^{\infty} g''_n(x) \) is uniformly convergent on \( I \).

Then \( \sum_{n=0}^{\infty} g_n(x) \) is twice differentiable on \( I \) and for each \( x \in I \):

\[
\left( \sum_{n=0}^{\infty} g_n(x) \right)' = \sum_{n=0}^{\infty} g'_n(x)
\]

\[
\left( \sum_{n=0}^{\infty} g_n(x) \right)'' = \sum_{n=0}^{\infty} g''_n(x)
\]

**Proof.** The hypotheses of the previous proposition are satisfied for \( g'_n(x) \) so we may conclude from it that \( \sum_{n=0}^{\infty} g'_n(x) \) is differentiable and its derivative is given by term-by-term differentiation. We also know that \( \sum_{n=0}^{\infty} g''_n(x) \) converges uniformly on bounded subintervals of \( I \). This allows us to again apply the previous proposition to show that \( \sum_{n=0}^{\infty} g''_n(x) \) is differentiable on \( I \) and its derivative is given by term-by-term differentiation. Therefore, \( \sum_{n=0}^{\infty} g_n(x) \) is twice differentiable. \( \square \)

Obviously, the reasoning may be extended to obtain results for higher order derivatives.

**3.9. Heat Equation on a Ring.** Consider a very thin ring, a circle, of circumference \( 2\pi \) whose lateral sides are insulated so that no heat may escape. If we parameterize length along the ring by \( x \in [-\pi, \pi] \) then the temperature \( u(x, t) \) satisfies the initial boundary value problem:

\[
\begin{align*}
    u_t &= au_{xx}, \quad -\pi < x < \pi, \ t > 0 \\
    u(-\pi, t) &= u(\pi, t), \quad t > 0 \\
    u_x(-\pi, t) &= u_x(\pi, t), \quad t > 0 \\
    u(x, 0) &= f(x), \quad -\pi < x < \pi
\end{align*}
\] (24)
The boundary conditions express the fact that there is no boundary to a ring so the temperature and heat flux must agree at \( x = -\pi \) and \( x = \pi \) since they represent the same point of the ring!

Looking for solutions of the form \( u = X(x)T(t) \) we find on inserting this into the PDE that

\[
\frac{T'(t)}{aT(t)} = \frac{X''(x)}{X(x)} = K, \quad \text{a constant}
\]

and so \( X \) satisfies

\[
X'' - KX = 0, \quad -\pi < x < \pi
\]

\[
X(-\pi) = X(\pi)
\]

\[
X'(-\pi) = X'(\pi)
\]

and \( T \) satisfies

\[
T'' = aKT
\]

We can show that \( K \leq 0 \) by the following argument. Multiply the differential equation for \( X \) by \( X \), integrate over \([-\pi, \pi]\) to get

\[
K \int_{-\pi}^{\pi} X^2 \, dx = -\int_{-\pi}^{\pi} (X')^2 \, dx
\]

Since \( X(x) \) cannot be identically zero for all \( x \), we see that \( K \leq 0 \). Set \( K = -\lambda^2 \), \( \lambda \geq 0 \). Then

\[
X = a \cos(\lambda x) + b \sin(\lambda x)
\]

and forcing \( X \) to satisfy the boundary conditions leads to:

\[
2b \sin(\lambda \pi) = 0
\]

\[
2a \lambda \sin(\lambda \pi) = 0
\]

If \( \sin(\lambda \pi) \neq 0 \), then, since \( \lambda \neq 0 \), we must have \( a = b = 0 \) which implies \( X \equiv 0 \) - an uninteresting result (\( u = XT = 0! \)). We conclude that

\[
\sin(\lambda \pi) = 0 \Rightarrow \lambda = n \in \{0, 1, 2, \cdots \}
\]

and hence

\[
X_0 = a_0/2, \quad X_n = a_n \cos(nx) + b_n \sin(nx), \quad n \geq 1, \quad T_0 = 1, \quad T_n = e^{-an^2t}
\]

and by superposition we have

\[
u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-an^2t}[a_n \cos(nx) + b_n \sin(nx)]
\]

Setting \( t = 0 \)

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]
\]
3. ADDITIONAL NOTES

Extending \( f(x) \) as a \( 2\pi \)-periodic function to all \( x \) by translation, we see that

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx
\]

Now we must verify that (25) solves our initial boundary value problem. Since each \( u_n \) is a solution of the PDE, \( u \) will be a solution provided the series converges and interchange of differentiation with the infinite sum is verified. For the latter, we see that both \( u_t \) and \( u_{xx} \) involve virtually the same series. We also note that the decaying exponential term should work in favor of convergence for \( t > 0 \). Fix \( x \in [-\pi, \pi] \) and \( \bar{t} > 0 \) arbitrarily small and consider (25) as a function of \( t \in I = \{ t \geq \bar{t} \} \). We are going to apply Proposition 2-3, Chapter 3 to verify that \( u \) is differentiable with respect to \( t \) and that its derivative is:

\[
(27) \quad u_t(x, t) = -a \sum_{n=1}^{\infty} n^2 e^{-an^2 \bar{t}} [a_n \sin(nx) + b_n \cos(nx)]
\]

According to Proposition 2-3, we must show that the above series converges uniformly on \( I \) and that the series (25) for \( u \) converges, say at \( t = \bar{t} \). The series obtained from the \( u_t \) series by taking absolute values is dominated by the series

\[
\sum_{n=1}^{\infty} n^2 e^{-an^2 \bar{t}} |a_n|, \quad \sum_{n=1}^{\infty} n^2 e^{-an^2 \bar{t}} |b_n|
\]

for \( t \in I \). Observe that the similar series corresponding to \( u \) (which does not have the factor \( n^2 \)) will be dominated by the series above as well! So we just have to find conditions for these two series to converge. The effect of the decaying exponential is so strong that we merely have to assume that there is some \( M > 0 \) so that

\[
(28) \quad |a_n|, |b_n| \leq M, \quad n \geq 1
\]
If this is the case then we may focus on

\[
\sum_{n=1}^{\infty} n^2 e^{-an^2\bar{t}} = \sum_{n=1}^{\infty} n^2 e^{-an^2\bar{t}}/2 e^{-an^2\bar{t}/2}
\]

\[
\leq L \sum_{n=1}^{\infty} (e^{-a\bar{t}/2}) n^2
\]

\[
\leq L \sum_{l=1}^{\infty} (e^{-a\bar{t}/2})^l
\]

\[
= L \frac{e^{-a\bar{t}/2}}{1 - e^{-a\bar{t}/2}} < \infty
\]

where we have used the fact that

\[ z \to z e^{-az\bar{t}/2} \]

is bounded by the constant \( L = \frac{2e^{-1}}{a\bar{t}} \) for \( 0 \leq z < \infty \) to conclude that

\[ n^2 e^{-an^2\bar{t}/2} \leq L. \]

Proposition 2-3 and our calculations above show that \( u \) is differentiable on \( I \) and \( u_t \) is given by (27) on \( I \). Since \( \bar{t} > 0 \) was arbitrary, we conclude that \( u_t \) is given by (27) for every \( t > 0 \).

An entirely parallel argument using the extension of Proposition 2-3 to second derivatives is required to show that \( u_{xx} \) exists and is given by the formula (27) but without the factor \( a \). With \( t > 0 \) now fixed, we must show that the series for \( u \) and \( u_x \) converge for some \( x \) and that for \( u_{xx} \) converges uniformly for \( x \in [-\pi, \pi] \). The same argument as for \( u_t \) shows the series for \( u_{xx} \) converges uniformly and absolutely on \( [-\pi, \pi] \) and that series dominates the series for \( u \) and \( u_x \) so they also converge uniformly on \( [-\pi, \pi] \).

We must also verify that \( u \) satisfies the initial conditions (26). But this holds by Dirichlet’s theorem if the 2\( \pi \)-periodic extension of \( f \) is continuous and piecewise smooth. This condition will automatically imply that (28) holds.

In summary, we have

**Theorem 1.** Let \( f \) be piecewise smooth and continuous on \( [-\pi, \pi] \) and satisfy

\[
f(-\pi) = f(\pi)
\]

Then (25) is a solution of (24). However, \( u(x, t) \) need not be continuous at points \( (x, 0) \), \( -\pi \leq x \leq \pi \). If \( f' \) is continuous and satisfies

\[
f'(-\pi) = f'(\pi)
\]
and \( f'' \) is piecewise continuous on \([-\pi, \pi]\) then \( u(x, t) \) is continuous for all \((x, t) \in [-\pi, \pi] \times [0, \infty)\).

**Proof.** Let \( \hat{f} \) be the \( 2\pi \)-periodic extension of \( f \). Then \( \hat{f} \) is piecewise smooth because \( f \) is. Since \( f \) is continuous, the Fourier series for \( \hat{f} \) converges to \( \hat{f}(x) = f(x) \) for each \( x \in (-\pi, \pi) \) by Dirichlet’s Theorem. At \( x = \pi \), it converges to \( \hat{f}(\pi) = f(\pi) \). Thus the Fourier series converges to \( f(x) \) at \( x = \pi \) by Dirichlet’s Theorem.

\( u(x, t) \) denotes the population density. The meaning is that \( \int_a^b u(x, t) \, dx \) is the number of individuals in \( a < x < b \) at time \( t \). If the organisms making up the population move randomly about their habitat, then it is reasonable to model this by the equation

\[
\begin{align*}
  u_t &= du_{xx} + ru \\
  u(0, t) &= u(L, t) = 0 \\
  u(x, 0) &= f(x)
\end{align*}
\]
The boundary conditions say that the boundary is lethal. Organisms cannot survive there. The diffusion constant \( d > 0 \) reflects the rate of the random movement.

**Exercise 0.7.** (a) Use the separation of variables technique or a change of variables \( v = e^{\lambda t} u \) to find a formal solution of (30). How big must the habitat be in order that the population can survive? What might this say about the destruction of habitat caused by humans or how big to design wild life refuges.

(b) Show that if \( f(x) \geq 0, \, 0 \leq x \leq L \), then \( u(x, t) \geq 0 \) for \( 0 \leq x \leq L \) and \( t > 0 \). Hint: Apply the Maximum Principle to \( v(x, t) = u(x, t)e^{\lambda t} \) for some \( \lambda \), or use Problem 2, section 5.3.

(c) If we change the boundary conditions to \( u_x(0, t) = 0 = u_x(L, t) \), then the organisms cannot leave the habitat (no flux!). Find a formal solution of this problem. Integrate the equation with respect to \( x \) to show that the total populations satisfies

\[
\int_0^L u(x, t) dx = e^{rt} \int_0^L f(x) dx
\]

**3.11. Fundamental Solution of the Heat Equation.** We seek a positive, integrable solution of the heat equation

\[
 u_t = au_{xx}, \quad -\infty < x < \infty, \quad t > 0
\]

which captures the dynamics following a drop of dye at \( x = 0 \) at \( t = 0 \). This solution should, by symmetry, be an even function of \( x \), have a single peak near \( x = 0 \) which we expect to slowly fall and widen as time progresses. It should go to zero at infinity fast enough that the following calculations hold. Integrate the equation to obtain

\[
\frac{d}{dt} \int_{-\infty}^{\infty} u(x, t) dx = \int_{-\infty}^{\infty} u_t(x, t) dx
\]

\[
= a \int_{-\infty}^{\infty} u_{xx}(x, t) dx
\]

\[
= au_x(x, t)|_{x=+\infty}^{x=-\infty}
\]

\[
= 0
\]

Thus,

\[
\int_{-\infty}^{\infty} u(x, t) dx = \text{constant}
\]

Observe that from the equation, \( a \) has units length^2/time so that

\[
z = \frac{x^2}{at}
\]
is dimensionless. It seems natural to try a solution of the form
\begin{equation}
(31) \quad u(x, t) = w(t)h(z) = w(t)h\left(\frac{x^2}{at}\right)
\end{equation}
where \(w(t)\) is chosen so that
\[
\int_{-\infty}^{\infty} u(x, t)dx = w(t) \int_{-\infty}^{\infty} h\left(\frac{x^2}{at}\right)dx = \text{constant, } t > 0
\]
We need to find \(w\) and \(h\). Notice that:
\[
\int_{-\infty}^{\infty} h\left(\frac{x^2}{at}\right)dx = 2 \int_{0}^{\infty} h\left(\frac{x^2}{at}\right)dx = \int_{0}^{\infty} h(z)(at)^{1/2} \frac{dz}{z^{1/2}} = (at)^{1/2} \int_{0}^{\infty} h(z) \frac{dz}{z^{1/2}}
\]
Therefore, we should choose \(w(t) = \frac{1}{(at)^{1/2}}\). Hence, we seek a solution of the form
\begin{equation}
(32) \quad u(x, t) = \frac{1}{(at)^{1/2}} h\left(\frac{x^2}{at}\right)
\end{equation}
where only \(h = h(z)\) needs to be determined. We have
\[
u_x = \frac{2x}{(at)^{3/2}} h'(z)
\]
\[
u_{xx} = \frac{4x^2}{(at)^{5/2}} h''(z) + \frac{2}{(at)^{3/2}} h'(z) = \frac{4}{(at)^{3/2}} z h''(z) + \frac{2}{(at)^{3/2}} h'(z)
\]
\[
u_t = -\frac{1}{2} a^{-1/2} t^{-3/2} h(z) - \frac{x^2}{t(at)^{3/2}} h'(z) = -\frac{1}{2} a^{-1/2} t^{-3/2} h(z) - \frac{a}{(at)^{3/2}} z h'(z)
\]
If we stick the last two terms into the heat equation and then multiply through by \((at)^{3/2} / a\), we get
\begin{equation}
(33) \quad 4zh''(z) + (z + 2)h'(z) + (1/2)h(z) = 0, \quad z > 0
\end{equation}
Try
\[
h(z) = e^{-rz}
\]
Why not? It has worked before! We could get lucky! We find, on dividing out the common factor \(e^{-rz}\), that
\[
4zr^2 - (z + 2)r + 1/2 = z(4r^2 - r) - 2r + 1/2 = 0, \quad \forall z > 0
\]
Clearly, we must have
\[
r(4r - 1) = 0, \quad \& \quad 0 = -2r + 1/2
\]
Fortunately, $r = 1/4$ satisfies both. In summary

$$(34) \quad u(x, t) = \frac{1}{(at)^{1/2}} e^{-\frac{x^2}{4at}}$$

is a positive, integrable solution of the heat equation which has a constant integral over $-\infty < x < \infty$, independent of $t > 0$.

Equation (34) reminds me of the normal distribution from statistics. Its density function, the infamous bell-curve, is given by

$$(35) \quad n(x, \sigma) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-\frac{x^2}{2\sigma^2}}$$

where $\sigma > 0$ is the standard deviation and $\sigma^2$ is the variance. $n$ is a probability density function so it has the property that

$$\int_{-\infty}^{\infty} n(x, \sigma) dx = 1$$

From statistics, I recall that 68% of the area under the normal density curve lies within one standard deviation of the mean $x = 0$:

$$\int_{-\sigma}^{\sigma} n(x, \sigma) dx = .68$$

Comparing (34) and (35), we see that our $u$ is essentially $n$ with standard deviation $\sigma = (2at)^{1/2}$. This suggests multiplying our $u$ by $1/(4\pi)^{1/2}$, which is again a positive solution of the heat equation:

$$(36) \quad u(x, t) = \frac{1}{(4\pi at)^{1/2}} e^{-\frac{x^2}{4at}}$$

This function has total integral equal one:

$$\int_{-\infty}^{\infty} u(x, t) dx = 1$$

It is called the fundamental solution of the heat equation.

The fundamental solution approaches a singularity as $t \to 0+$ since its value at $x = 0$ blows up: $u(0, t) \to \infty$. In fact, it approximates the famous Dirac “function” $\delta(x)$ as $t \to 0+$:

$$u(x, t) \to \delta(x), \quad t \to 0+$$

Recall that $\delta(x)$ is supposed to have the properties:

1. $\delta(x) = 0$ for $x \neq 0$.
2. If $a < 0 < b$ and $f$ is a continuous function then $\int_{a}^{b} f(x) \delta(x) dx = f(0)$. 

If \( u(x, t) \) solves the heat equation, so does \( u(x - y, t) \) for any fixed \( y \). In the case of our fundamental solution, \( u(x - y, t) \to \delta(x - y) \) as \( t \to 0^+ \). If \( f \) is any bounded continuous function, then superposition suggests that

\[
v(x, t) = \int_{-\infty}^{\infty} f(y)u(x - y, t)dy
\]
is also a solution (differentiate under integral) and

\[
v(x, t) = \int_{-\infty}^{\infty} f(y)u(x - y, t)dy \to \int_{-\infty}^{\infty} f(y)\delta(x - y)dy = f(x), \ t \to 0^+
\]

It can be shown that

\[
v(x, t) = \frac{1}{(at)^{1/2}} \int_{-\infty}^{\infty} f(y) \frac{1}{(at)^{1/2}} e^{-\frac{(x-y)^2}{4at}} dy
\]
satisfies the initial value problem

\[
\begin{align*}
v_t & = av_{xx}, \ -\infty < x < \infty, \ t > 0 \\
v(x, 0) & = f(x), \ -\infty < x < \infty
\end{align*}
\]

### 3.12. Eigenvalue Problem for the Laplacian

Let \( D \) be a normal domain in \( \mathbb{R}^n \). For the Dirichlet boundary condition, the problem is to find eigenvector-eigenvalue pairs \( (u, \lambda) \), where \( u \) is not the zero function, which satisfy

\[
\begin{align*}
0 & = \Delta u + \lambda u, \ x \in D \\
0 & = u(x), \ x \in B
\end{align*}
\]
As a familiar example, if $n = 1$, $D = (0, L)$, the eigenvalue problem becomes

$$
0 = u'' + \lambda u, \quad x \in (0, L)
$$
$$
0 = u(0) = u(L)
$$

The eigenvalues and eigenfunctions are

$$
\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad u_n = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots
$$

We use Green’s Identities to show that $\lambda$ must be positive. Suppose that $u$ is an eigenfunction corresponding to a real eigenvalue $\lambda$. Multiply the eigenvalue equation by $u$ and integrate to get

$$
0 = \int_D (u\nabla u + \lambda u^2) dx
$$
$$
= -\int_D |\nabla u|^2 dx + \lambda \int_D u^2 dx
$$

where we have used Green’s first identity. Since $u$ is not the zero function, $\int_D u^2 dx \neq 0$ so

$$
\lambda = \int_D |\nabla u|^2 dx / \int_D u^2 dx \geq 0
$$

If $\int_D |\nabla u|^2 dx = 0$ then $\nabla u = 0$ in $D$ which implies that $u$ is a constant. As $u = 0$ on $B$, it would follow that $u = 0$ in $D$ contradicting that $u$ is not the zero function. Thus we conclude that $\lambda > 0$.

Now suppose that $\lambda_1$ and $\lambda_2$ are distinct eigenvalues with corresponding eigenfunctions $u_1$ and $u_2$. We claim that

$$
\int_D u_1(x)u_2(x) dx = 0
$$

Indeed, apply Green’s second identity to $u_1$ and $u_2$ to conclude

$$
0 = \int_D (u_1\nabla u_2 - u_2\nabla u_1) dx
$$
$$
= \int_D -\lambda_2 u_1 u_2 + \lambda_1 u_1 u_2 dx
$$
$$
= (\lambda_1 - \lambda_2) \int_D u_1 u_2 dx
$$
If $D = (0, a) \times (0, b) \subset \mathbb{R}^2$ where $a, b > 0$, then separation of variables gives the eigenvalues and eigenfunctions as

$$
\lambda_{nm} = \pi^2 \left( \frac{n^2}{a^2} + \frac{m^2}{b^2} \right),
$$

$$
u_{nm} = \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi y}{b} \right), \ n, m = 1, 2, 3, \ldots
$$

Clearly, this extends in the obvious way to the case that $D = (0, a) \times (0, b) \times (0, c) \subset \mathbb{R}^3$.

3.13. Exercise 0.2. Let $f : (a, b) \rightarrow \mathbb{R}$ be a real-valued function such that $f'(x)$ exists for $x \in (a, b)$, $f'(x_0) = 0$ for some $x_0$, and suppose that $f''(x_0) > 0$.

Since

$$f''(x_0) = \lim_{x \rightarrow x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$$

Given $\epsilon > 0$ we can find $\delta > 0$ such that

$$-\epsilon < \frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0) < \epsilon$$

provided $-\delta < x - x_0 < \delta$. If we choose $\epsilon = \frac{f''(x_0)}{2}$, for example, then there is a corresponding $\delta$ as above. Since $f'(x_0) = 0$, we conclude that

$$0 < \frac{f''(x_0)}{2} < \frac{f'(x) - f'(x_0)}{x - x_0}$$

provided $-\delta < x - x_0 < \delta$. It follows that

$$f'(x) > 0, \ x_0 < x < x_0 + \delta.$$

Now, if $x_0 < x < x_0 + \delta$, the Mean Value Theorem implies there exists $c$, depending on $x$, such that $x_0 < c < x$ and

$$f(x) - f(x_0) = f'(c) \left( x - x_0 \right)$$

Since $f'(c) > 0$, we have proved that

$$f(x) > f(x_0), \ x_0 < x < x_0 + \delta$$

Now suppose that a differentiable function $f$ has a local maximum at $x_0$ and that $f''(x_0)$ exists. Note that we do not assume that $f''(x)$ exists at any other point! We know that $f'(x_0) = 0$ at an extrema. If $f''(x_0) > 0$, then our argument above shows that $f(x) > f(x_0)$ for $x_0 < x < x_0 + \delta$. This is incompatible with our assumption that $x_0$ is a local maximum. Therefore, we conclude that $f''(x_0) \leq 0$. 
Finally, suppose that $u$ is a differentiable function on an open set $D \subset \mathbb{R}^3$ which attains a local maximum at $x_0 = (x_0^1, x_0^2, x_0^3)$ and that $\frac{\partial^2 u}{\partial x_j^2}(x_0), \ j = 1, 2, 3$, exist. Since $x_0$ is an extrema, we have

$$\frac{\partial u}{\partial x_j}(x_0) = 0, \ j = 1, 2, 3$$

These hypotheses imply that $f(x) = u(x, x_0^2, x_0^3)$ has a local maximum at $x = x_0^1$ and that

$$f'(x_0^1) = \frac{\partial u}{\partial x_1}(x_0) = 0, \ f''(x_0^1) = \frac{\partial^2 u}{\partial x_1^2}(x_0)$$

Our results above imply that $f''(x_0^1) \leq 0$ so we conclude that

$$\frac{\partial^2 u}{\partial x_1^2}(x_0) \leq 0$$

Similarly, for the other variables we get

$$\frac{\partial^2 u}{\partial x_j^2}(x_0) \leq 0, \ j = 1, 2, 3$$

so

$$\Delta u(x_0) \leq 0.$$  

Consider our population with density function $u(x, t)$ which satisfies:

$$u_t = d\Delta u + ru, \ x \in D, \ t > 0$$
$$u(x, t) = 0, \ x \in B, \ t > 0$$
$$u(x, 0) = f(x), \ x \in D$$

Here, we imagine $D$ as a bounded domain in $\mathbb{R}^2$ (a Petri dish?) or $\mathbb{R}^3$.

Letting $v = e^{-rt}u$ we see that $v$ satisfies the heat equation with the same initial conditions and boundary conditions. From this fact and the maximum principle, we have the following conclusions:

1. If $f(x) \geq 0, \ x \in D$, then $u(x, t) \geq 0, \ x \in D, t > 0$.
2. If $f(x) \leq g(x), \ x \in D$ and $u$ is the solution with initial data $f$ and $u^*$ is the solution with initial data $g$, then $u(x, t) \leq u^*(x, t), \ x \in D, t > 0$.  


3. ADDITIONAL NOTES

The smallest eigenvalue $\lambda_1 > 0$ of the Laplacian and corresponding eigenfunction $u_1(x)$ will play a key role.

$$
0 = \triangle u + \lambda u, \quad x \in D \\
u(x) = 0, \quad x \in B
$$

It is a fact that $u_1(x) > 0$, $x \in D$ (but not on $B$!). For example, if $D = (0, a) \times (0, b)$ then

$$
\lambda_1 = \pi^2 \left( \frac{1}{a^2} + \frac{1}{b^2} \right), \quad u_1(x, y) = \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right)
$$

It is easy to check that for any $c > 0$

$$
u(x, t) = ce^{(r-\lambda_1 d)t} u_1(x)
$$

is a positive solution.

This solution grows if $r - \lambda_1 d > 0$ and decays if $r - \lambda_1 d < 0$. In the case of the square $D$ described above we find that

$$
r - \lambda_1 d > 0 \iff \sqrt{\frac{r}{d}} > \pi \frac{\sqrt{a^2 + b^2}}{ab}
$$