First-fit Coloring on Interval Graphs Has Performance Ratio at Least 5

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Abstract. First-fit is the online graph coloring algorithm that considers vertices one at a time in some order and assigns each vertex the least positive integer not used already on a neighbor. The maximum number of colors used by first-fit on graph \( G \) over all vertex orders is denoted \( \chi_{FF}(G) \).

The exact value of \( R := \sup_G \frac{\chi_{FF}(G)}{\omega(G)} \) over interval graphs \( G \) is unknown. Pemmaraju, Raman, and Varadarajan proved \( R \leq 10 \), and this can be improved to 8. Witsenhausen and Chrobak and Šlusarek showed \( R \geq 4 \), and Šlusarek improved this to 4.45. We prove \( R \geq 5 \).

Key words: graph coloring, online algorithm, first-fit, interval graph.

1. Introduction

Every coloring\(^1\) of graph \( G \) has at least \( \omega(G) \) colors.\(^2\) Every interval\(^3\) graph\(^4\) \( G \) has a coloring of just \( \omega(G) \) colors. One can be constructed by ordering intervals by left end and coloring by first fit.

In some applications, however, we begin assigning colors before the whole graph is seen. A coloring algorithm is online if, given graph \( G \) on vertices \( v_1, \ldots, v_n \), it assigns (irrevocably) for each \( k \in \{1, \ldots, n\} \) a color to \( v_k \) that depends only on the subgraph of \( G \) induced by seen vertices \( \{v_j \mid 1 \leq j \leq k\} \). For example, the first-fit algorithm produces coloring \( f \) as follows: for \( k \) from 1 to \( n \), let \( f(v_k) \) be the least positive integer available, i.e. not a member of \( \{f(v_j) \mid 1 \leq j \leq k \text{ and } v_jv_k \text{ is an edge}\} \).

The coloring produced by this algorithm depends on vertex order. For example, Figures 1 and 2 show first-fit colorings of the 4-vertex path.

First-fit may be the first algorithm that comes to mind when the goal is to use few colors. How wasteful is it in the worst case? Let \( \chi_{FF}(G) \) be the maximum over all vertex orders of the number of colors it uses on graph \( G \). For example, \( \chi_{FF}(P_4) = 3 \).

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\(^1\)A coloring of graph \( (V,E) \) is a function \( f : V \to \mathbb{Z} \) where \( f(u) \neq f(v) \) for all \( uv \in E \).

\(^2\)The clique size of \( G \) is the number \( \omega(G) \) of vertices in a largest complete subgraph of \( G \). We consider only finite graphs.

\(^3\)An interval is a convex set of real numbers.

\(^4\)If each vertex \( v \) of \( G \) is associated with an interval \( I_v \), and edges of \( G \) are precisely the vertex pairs \( uv \) where \( I_u \cap I_v \neq \emptyset \), then \( G \) is an interval graph.
Kierstead [5] showed that $\chi_{FF}(G) \leq 40\omega(G)$ for every interval graph $G$. This was improved to $25.72\omega(G)$ by Kierstead and Qin [6], then $10\omega(G)$ by Pemmaraju, Raman, and Varadarajan [8]. Brightwell, Kierstead, and Trotter [1] and Narayanaswamy and Subhash Babu [7] observed that the technique in [8] yields $8\omega(G)$. These results provide bounds above the worst-case performance ratio of the first-fit algorithm on interval graphs,

$$R := \sup \left\{ \frac{\chi_{FF}(G)}{\omega(G)} \middle| G \text{ is a nonempty interval graph} \right\}.$$ 

So $1 \leq R \leq 8$. We want to know the exact value of $R$. This motivates a search for lower bounds.

Already $1.5 \leq R$ due to the example $P_4$. Variations on this example yield improvements, and this is the structure of the present paper: we recapitulate results of earlier authors in Section 2, generalize their constructions in Section 3, and tune the new construction to get $5 \leq R$ in Section 4. In Section 5 we show that this result is best possible in some sense.

2. Walls and caps

A bound below $R$ follows from a given interval graph and vertex order. Gyárfás and Lehel [4] defined a wall to be essentially a graph and vertex ordering. Rather than a linear vertex order, we will give a coloring and claim that it is the result of the first-fit algorithm in some order. For our purpose, pair $(G, f)$ is an $r$-wall if:\footnote{Some common notation: let $N(v)$ be the set of neighbors of vertex $v$, and $N[v] := N(v) \cup \{v\}$. When $f$ is a function on domain $V$, and $U \subseteq V$, we denote the image of $U$ by $f(U)$.}

1. $G$ is an interval graph\footnote{Whenever a wall is given, an interval representation is included.} on vertex set $V \neq \emptyset$;
2. $f : V \to \{1, 2, \ldots\}$ is a coloring of $G$;
3. $f(N[v]) \supseteq \{1, \ldots, f(v)\}$ for all $v \in V$ (support); and
4. $|f(V)| \geq r\omega(G)$.

For example, a 2-wall appears in Figure 3. We think of each vertex as a brick in a wall. The coloring of Condition 2 effectively groups vertices into levels (color classes). Condition 3 means that each brick meets some other in each level below.

Proposition 1. If an $r$-wall exists, then $r \leq R$. 

![Figure 1. First-fit uses 2 colors on $P_4 = v_1v_2v_3v_4$](image1)

![Figure 2. First-fit uses 3 colors on $P_4 = v_1v_3v_4v_2$](image2)
Proof. Given \( r \)-wall \((G, f)\) where \( V \) is the vertex set of \( G \), let \( v_1, \ldots, v_n \) be an enumeration of \( V \) that increases in color, i.e. \( f(v_i) \leq f(v_j) \) whenever \( 1 \leq i < j \leq n \). Then the first-fit algorithm (applied to this order) assigns color \( f(v) \) to \( v \) for each \( v \in V \). In particular, \( \chi_{FF}(G) \geq f(v_n) \geq r \omega(G) \). \( \Box \)

So \( 2 \leq R \) by Figure 3. Also consider the sequence \( W_0, W_1, \ldots \) of walls depicted in Figure 4, where \( W_0 \) is the 1-vertex wall (and \( W_1 \) is the previous example). So \( 3 \leq R \). We generalize this arrangement of new intervals atop previously constructed walls (as in Figure 4) by defining a wall-like object called a cap.

Let \( G \) be a (finite) nonempty interval graph on vertex set \( V \) with interval representation \( v \mapsto I_v \). Let \( f : V \rightarrow \mathbb{Z} \) be a coloring\(^7\) of \( G \). The remaining conditions (defined precisely below) are illustrated in Figure 5:

1. each interval \( I_v \) has empty space below it for a previously constructed wall\(^8\) \( W_v \);
2. few intervals are above \( W_v \); and
3. each level between the levels of \( I_v \) and \( W_v \) contains a cap interval that meets \( I_v \).

Suppose for each \( v \in V \) there is an interval \( J_v \subseteq I_v \) where \((\forall u, v \in V)[J_u \cap J_v \neq \emptyset \rightarrow J_u = J_v] \). We denote the colors of intervals \( I_u \) that meet \( J_v \) by

\[ C_v := f(\{u \in V \mid I_u \cap J_v \neq \emptyset\}). \]

If for each \( v \in V \) there is a color \( c_v \in \mathbb{Z} \) where:

\(^{7}\)Cap colors need not be positive. Colors will be translated anyway when a cap is placed atop previously constructed walls. Also, caps are built from the top down. It may not be clear at first how many colors are needed. So we use 0 for the top level and build the cap downward using negative integers.

\(^{8}\)Unlike the wall sequence of Figure 4, the subscript \( v \) in \( W_v \) is the vertex that wall \( W_v \) is intended to support. Walls of many sizes may be needed to construct the next wall in a sequence. However, usually there will be many vertex pairs \( u \neq v \) with \( W_u = W_v \).
Proposition 2. If an $r$-cap exists, then $r \leq R$; specifically, there is a constant $b$ so that for all $k \in \{0,1,\ldots\}$, there is a wall of clique size $k$ with at least $rk - b$ colors.

Proof. Argue by induction on $k$. This is the strong form of induction: unlike the construction of Figure 4, old walls of various sizes may be used in a single inductive step (see Figure 6).

The inductive step is to construct a wall of clique size $k$ (for $k$ larger than the base cases). Let $v$ be a cap vertex. Its old wall $W_v$ will have above it no clique on more than $\floor{-c_v/r}$ vertices. So to meet the goal of the construction, $W_v$ should have clique size (at most)
Figure 7. The 4-cap of Chrobak and Šlusarek

By the inductive hypothesis, such a wall exists with at least \( r(k - \left\lfloor \frac{-c_v}{r} \right\rfloor) - b \geq rk + c_v - b \) colors. It can be squeezed horizontally into the interval \( J_v \). Thus the cap coloring \( f \) may be extended to \( W_v \) so as to satisfy the support condition. After this is done for all vertices, colors are translated by a common number so that 1 is the least color used. The result is a wall of clique size \( k \) and at least \( rk - b \) colors.

It remains to deal with base cases and the value of \( b \). For \( k < |f(V)| \) (the number of colors in the cap), a wall on \( k \) vertices with constant interval representation \( v \mapsto [0,1] \) suffices. Let \( b = r|f(V)| \).

Witsenhausen [12] proved \( 4 \leq R \), as did Chrobak and Šlusarek [2], who defined\(^9\) the 4-cap of Figure 7. It is constructed from the top down by the following considerations:

1. Twin 1-high boxes\(^10\) go at the top. The rest of the cap is the same left and right. We describe one side. A cone goes below the 1-high box at depth suited to the target ratio \( r = 4 \). The 1-high box still requires support in 2 colors below.
2. A 2-high box fills the gap. A cone goes below it at proper depth. A gap of 4 colors remains. It is possible to insert a 1-high box (and cone) on the inside, leaving a gap of only 3 on the outside.

So by Proposition 2 we have \( 4 \leq R \).

How may we improve this result? Perhaps we should try the procedure with \( r = 5 \). Let us record the sequence of box heights along the outer course. For \( r = 4 \) it is\(^11\) 1, 2, 3, 4, 4.

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\(^9\)A number inside a box indicates vertices with a common interval and consecutive colors.

\(^10\)A cap is nonempty, so a box is needed. One is everyone’s favorite positive number. A twin is not essential to the case \( r = 4 \), but our generalization below does rely on this bilateral symmetry.

\(^11\)We omit final box height 1 because that box is just a copy of the last inner box.
The procedure halts because the last number does not exceed its predecessor. For \( r = 5 \) the sequence is 1, 3, 8, 22, 61, 170, 475, 1329, 3721, 10422, 29196, \ldots. It seems to increase without end. Does it? If so, why?

Crucially, the sequence \( u_1, u_2, \ldots \) obeys a (linear) recurrence. Depths of two (adjacent) cones as in Figure 8 are related by

\[
ru_n - (u_n - 2 - u_{n-3}) + u_n + (u_{n-1} - u_{n-2}) = ru_{n-1},
\]

so heights of boxes of the outer course are given by the sequence

\[
\begin{align*}
  u_0 &= u_1 = 1 \\
  u_2 &= r - 2 \\
  u_n &= (r - 1)u_{n-1} - (r - 2)u_{n-2} - u_{n-3}
\end{align*}
\]

for \( n \geq 3 \). What determines the eventual behavior of such sequences? The discriminant

\[
D = -31 + 6r + 7r^2 - 6r^3 + r^4
\]

of the characteristic polynomial \( 1 - (r - 1)x + (r - 2)x^2 + x^3 \) of the recurrence is negative when \( r = 4 \) and positive when \( r = 5 \). As we will see when we study a more general construction, such differences have important effects on sequence behavior.

Meanwhile, the bound \( 4 \leq R \) may be improved by considering noninteger \( r \) as Šlusarek [10] did when he obtained \( 4.45 \leq R \). Observe that if each number in Figure 7 is doubled, say, the result is again a cap. Therefore we may use (temporarily) rational numbers instead of integers for box heights. \( D \) has a root

\[
r_+ = 1.5 + 0.5\sqrt{13 + 16\sqrt{2}} \approx 4.48.
\]

The best\(^{12}\) we can do with this (rational) generalization of the method of Chrobak and Šlusarek is \( r_+ \leq R \).

\(^{12}\)For \( r \in \{4, 5, \ldots\} \).

\(^{13}\)Of course, \( r_+ \) is irrational, but it can be approximated by rationals.
3. A new construction

In the previous section, our hope to obtain a lower bound \( 5 \leq R \) was dashed by an unhalting procedure.\(^{14}\) Yet a certain resource was unused. The cap grew only 1 box inward. We will enable it to grow further.

First a note on sequences. It is impossible by our method to construct a 5-cap directly (see Section 5). So you might imagine that we construct a 4.9-cap, then a 4.99-cap, and so on. Yes, but not until the proof of Theorem 12, and anyway there is no need for a specific sequence. Rather we will show that an \( r \)-cap exists for every \( r < 5 \). In the present section, \( r \) is fixed. When we describe a sequence, we mean a finite sequence of quasi-cap constructions, where a certain deficiency is reduced incrementally to 0.

A quasicap is a cap, except for the support condition. In our construction, only two 1-high boxes at the top of the cap are exempt from the support condition. The deficiency of these quasicaps is a vertical gap. Each of the two \((r-2)\)-high boxes (from the second step of the construction of Chrobak and Šlusarek) in the key position (the box marked \( \theta \) in Figure 9) next to a 1-high box is not tall enough to meet and support the 1-high box.

It remains to describe the deficiency-reducing step. Each quasicap in the sequence is constructed by the method of Chrobak and Šlusarek, but rather than put a single box on the inside,\(^{15}\) we put a (reflected\(^{16}\) shifted scaled) copy of part of the previous quasicap of the sequence. One such incremental step is illustrated in Figures 9 and 10.

The heights of boxes in the outer course are defined by the linear recurrence\(^{17}\)

\[
\begin{align*}
    u_0 &= u_1 = 1 \\
    u_2 &= \theta + \delta \\
    u_n &= (r - \theta)u_{n-1} - (r - 2\theta)u_{n-2} - \theta u_{n-3}
\end{align*}
\]

for \( n \geq 3 \). Here \( \theta \) is the height of the key box in a previously constructed\(^{18}\) quasicap. We raise this by \( \delta > 0 \) in the step from Figure 9 to Figure 10.

**Proposition 3.** Fix some rational \( r > 2 \). Suppose there is a quasicap with key box height \( \theta \) (henceforth a \( \theta \)-quasicap). If there exist \( \delta > 0 \) and integer \( N > 2 \) so that \( u_{N-1} \geq u_N \), then a \((\theta + \delta)\)-quasicap exists.

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\(^{14}\)When the cap-construction procedure does not halt, we have no cap to use in Proposition 2.

\(^{15}\)The final, outer box goes on the outside.

\(^{16}\)The final, outer copy is not reflected.

\(^{17}\)This generalizes the previous definition of \( u_n \), in which \((\theta, \delta) = (1, r - 3)\).

\(^{18}\)We might as well use the quasicap with greatest key box height.
Proof. We aim to ensure that each incremental step from a quasicap yields again a quasicap. This involves placing copies so that:

1. the sparse property of each box is kept, and
2. each copy goes precisely where needed to support a higher box.

The placement of copies can be done by horizontal and vertical affine transformations. Horizontal placement does not worry us. No cleverness is needed to move copies appropriately and shrink them enough that they don’t collide with other parts of the quasicap. We are concerned with vertical placement and its effect on the sparse condition.

We have noted already that we can expand (scale) caps vertically by an integral multiplicative factor. This applies also to quasicaps. And we may use rational multiplicative factors for intermediate operations. (At the end of the construction, integers are obtained.)

To transform the hatched portion of Figure 9 into a position (new position) where it may support the key box of Figure 10, say, our first operation is to scale the portion by a factor of \(u_2 - u_1\), the difference in height of the two boxes above the portion, giving a scaled copy. The next operation is to shift the scaled copy downward by a distance of \(ru_1\), giving a shifted scaled copy. In Figure 9 the hatched portion has a 1-high box over part of it and 0 over the rest. The scaled copy can “bear” (according to the support condition) a \((u_2 - u_1)\)-high box over part and 0 over the rest. We adapt the scaled copy to the new position by putting the equivalent of an additional \(u_1\)-high box over the whole scaled copy. The scaled copy must be shifted downward by a distance of \(ru_1\) to compensate (i.e. for sparse conditions to hold).

The result of the preceding paragraph is a shifted scaled copy (just copy). Let us verify that the copy is placed so as to support the new key box. It suffices to examine the location of the key box after scaling and shifting. Its bottom before scaling and shifting was at depth \(r\). Afterward it is at \((u_2 - u_1)r + ru_1 = ru_2\), which is the prescribed depth for the cone beneath the \(u_2\)-high box. The height of the key box before scaling and shifting was \(\theta\). Afterward it is \((u_2 - u_1)\theta\), while the gap it should cover spans the bottom of the \(u_3\)-high box.

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19Box height above the shifted scaled copy is \((u_2 - u_1)(1,0) + (u_1, u_1) = (u_2, u_1)\).
at depth \( ru_1 + u_3 \) and the top of the cone under the key box at depth \( ru_2 \), i.e. a gap of size
\[
ru_2 - ru_1 - u_3 = ru_2 - ru_1 - (r - \theta)u_2 + (r - 2\theta) + \theta = (u_2 - u_1)\theta.
\]
So the first copy, at least, is in the right place.

For each remaining copy, again we will verify that its key box has its bottom in the right place, and that it has the right height. The copy under boxes \( u_{n-1} \) and \( u_n \), where \( 2 < n < N \), is scaled by \( u_{n-1} - u_{n-2} \), then shifted down \( ru_{n-2} \) to accommodate the \( u_{n-2} \)-high box above. So the key box bottom is at depth \( (u_n - u_{n-1})r + ru_{n-1} = ru_n \), where it belongs. The key box after scaling and shifting has height \( (u_n - u_{n-1})\theta \). The gap it should cover has height that we calculate by a slight variation on Figure 8,
\[
ru_n - [ru_{n-1} - (u_{n-1} - u_{n-2})\theta + u_{n+1}].
\]
If we replace \( u_{n+1} \) using its recurrent definition, we see that these quantities are equal. We’re done.

It remains to deal with the copy under box \( u_N \) (and under no second box). But the analysis of case \( n = N - 1 \) in the previous paragraph applies here, too. (Differing only by a horizontal reflection and shift, the copies have equal depth, but while in case \( n = N - 1 \) we had boxes of height \( u_{N-2} \) and \( u_{N-1} \) above, here we have boxes of height \( 0 < u_{N-2} \) and \( u_N \leq u_{N-1} \). So the sparse conditions are not only satisfied, but may have some slack.) □

Clearly a 1-quasicap exists. And an \((r - 2)\)-quasicap is an \( r \)-cap. If we choose \( r \) and a finite sequence \( \delta_0, \delta_1, \ldots \) of positive rational improvements with \( \delta_0 + \delta_1 + \cdots = r - 3 \) that satisfy the hypothesis of Proposition 3, we get a lower bound \( r \leq R \). In the next section we show that such choices exist for all \( r < 5 \).

4. The special sequence

Sequence \( u_n \) of Section 3 is a linear homogeneous recursive sequence. So is its sequence of differences. The ordinary power series generating function
\[
f(x) = \sum_{n \geq 0} (u_{n+1} - u_n)x^n
\]
of the difference sequence is (cf. chapter 4 of [11])
\[
f(x) = \frac{p(x)}{q(x)},
\]
where \( q(x) \) is a polynomial whose coefficients are those of the recurrence, and \( p(x) \) is a polynomial of lesser degree related to boundary values of the sequence. Specifically,
\[
p(x) = x[(\theta + \delta)(1 - x) - 1]
\]
and
\[
q(x) = 1 - (r - \theta)x + (r - 2\theta)x^2 + \theta x^3
\]
(1)
\[
= 1 + rx(x - 1) + \theta(x - 1)^2
\]
(2)
\[
= (1 - x/\alpha)(1 - x/\beta)(1 - x/\gamma).
\]
(3)
Form (3) gives names $\alpha$, $\beta$, and $\gamma$ to the (complex) roots of $q(x)$. They affect the asymptotic behavior of the sequence. When they are distinct,

$$f(x) = \frac{A}{1-x/\alpha} + \frac{B}{1-x/\beta} + \frac{C}{1-x/\gamma}$$

and

$$u_{n+1} - u_n = A\alpha^{-n} + B\beta^{-n} + C\gamma^{-n}$$

for some complex numbers $A, B, C$. Using (3),

$$q(x)f(x) = A(1-x/\beta)(1-x/\gamma) + B(1-x/\alpha)(1-x/\gamma) + C(1-x/\alpha)(1-x/\beta).$$

The next observation applies in all cases that matter: one root, say $\alpha$, is real and unimportant. (Still complex are $\beta$ and $\gamma$.)

**Proposition 4.** When $1 \leq \theta \leq 0.5r \leq 3\theta$, we can assume $\alpha < -1$ and $0 < \beta\gamma < 1$.

**Proof.** The first conclusion follows from the intermediate value theorem and (2):

$$q(-1) = 1 + 2r - 4\theta > 0$$
$$q(-10) = 1 + 110r - 1210\theta < 0.$$

For the second conclusion, compare cubic terms of (1) and (3):

$$-\alpha\beta\gamma = \theta^{-1} \leq 1.$$

□

For the rest of the section we restrict $r$ and $\theta$:

$$4.999 \leq r \leq 5$$
$$1 \leq \theta \leq 2.2.$$

So Proposition 4 applies. This simplifies our calculations.

**Proposition 5.** When $\theta \leq 2.13$, we can assume $0 < \gamma < 0.56 < \beta < 1$.

**Proof.** By the intermediate value theorem. Using (2),

$$q(0) = q(1) = 1 > 0$$

and

$$q(0.56) \leq 1 - (4.999)(0.56)(0.44) + (2.13)(0.56)(0.44)^2 < 0.$$

□

**Proposition 6.** If $\alpha < -1 < 0 < \gamma < \beta < 1$, then $u_{n+1} - u_n \to -\infty$ when

$$\theta + \delta < \frac{1}{1-\gamma}.$$

**Proof.** In this case

$$u_{n+1} - u_n \sim C\gamma^{-n},$$

and the desired conclusion follows when $C < 0$. Comparing two expressions of $q(x)f(x)$,

$$x[(\theta + \delta)(1-x) - 1] = A(1-x/\beta)(1-x/\gamma) + B(1-x/\alpha)(1-x/\gamma) + C(1-x/\alpha)(1-x/\beta).$$
Substituting $\gamma$ for $x$,

$$\gamma[(\theta + \delta)(1 - \gamma) - 1] = C(1 - \gamma/\alpha)(1 - \gamma/\beta).$$

Because

$$0 < (1 - \gamma/\alpha)(1 - \gamma/\beta),$$

we have $C < 0$ when

$$(\theta + \delta)(1 - \gamma) - 1 < 0.$$

\[\square\]

**Proposition 7.** If $\theta \leq 2.13$, then $\gamma$ decreases strictly in $r$, and so does $\frac{1}{1 - \gamma} - \theta$.

*Proof.* Suppose

$$4.999 \leq r_0 < r_1 \leq 5.$$ 

Using (2) for $j \in \{0, 1\}$,

$$q_j(x) = 1 + r_j x(x - 1) + \theta x(x - 1)^2$$

has roots $0 < \gamma_j < \beta_j < 1$ by Proposition 5. When $\gamma_0 \leq x \leq \beta_0$, we have

$$0 \geq q_0(x) = 1 + r_0 x(x - 1) + \theta x(x - 1)^2$$

and

$$0 > (r_1 - r_0)x(x - 1).$$

Adding the last two inequalities,

$$0 > 1 + r_1 x(x - 1) + \theta x(x - 1)^2 = q_1(x).$$

Specifically, $q_1(\gamma_0) < 0$. Because $q_1(x) \geq 0$ for $x \in [0, \gamma_1] \cup [\beta_1, 1]$, it follows that $\gamma_1 < \gamma_0$. That is, $\gamma$ decreases in $r$. The second conclusion follows. 

\[\square\]

**Proposition 8.** If $\theta \leq 2.13$ and $r = 5$, then $\theta \leq \frac{1}{1 - \gamma}$.

*Proof.* Using (2),

$$q(1 - \theta^{-1}) = 1 + 5(1 - \theta^{-1})(-\theta^{-1}) + \theta(1 - \theta^{-1})\theta^{-2} = (1 - 2\theta^{-1})^2 \geq 0,$$

so $\gamma \geq 1 - \theta^{-1}$. 

\[\square\]

**Proposition 9.** If $\theta = 2.13$, then $\frac{1}{1 - \gamma} - \theta > 0.04$.

*Proof.* Proposition 7 implies that $\gamma$ is least when $r = 5$. Evaluate $q(0.54)$ there (using (2)) to obtain a lower bound for $\gamma$:

$$q(0.54) = 1 - (5)(0.54)(0.46) + (2.13)(0.54)(0.46)^2 > 0,$$

So $\gamma > 0.54$ and $\frac{1}{1 - \gamma} - \theta > 0.04$.

\[\square\]

The discriminant $D$ of $q(x)$ is (cf. pp. 95-102 of [9])

$$D = -27\theta^2 - 4\theta^3 + 6\theta^2 r + 6\theta r^2 + \theta^2 r^2 - 4r^3 - 2\theta r^3 + r^4,$$

and $D < 0$ if and only if $\text{Im}[\gamma]\text{Im}[\beta] \neq 0$.

**Proposition 10.** If $\theta = 2.15$, then $D < 0$. 

\[\text{11}\]
Proof. If $\theta \geq 2.1$, then $D$ increases in $r$:

$$\frac{dD}{dr} = 6\theta^2 + 12\theta r + 2\theta^2 r - 12r^2 - 6\theta r^2 + 4r^3$$

$$\geq 6(2.1)^2 + 12(2.1)(4.9) + 2(2.1)^2(4.9) - 12(5)^2 - 6(2.2)(5)^2 + 4(4.9)^3 > 0.$$ 

So $D$ is greatest when $r = 5$.

$$D = -27(2.15)^2 - 4(2.15)^3 + 6(2.15)^2(5) + 6(2.15)(5)^2 + (2.15)^2(5)^2 - 4(5)^3 - 2(2.15)(5)^3 + (5)^4 < 0.$$ 

□

Proposition 11. If $D < 0$ and $|\alpha| > 1 > |\beta| = |\gamma| > 0$, then $u_N \geq u_{N+1}$ for some $N > 0$.

Proof. In this case, (4) is equivalent to:

$$u_{n+1} - u_n = A\alpha^{-n} + 2\text{Re}[C\gamma^{-n}],$$

because Formula (5) for $A, B,$ and $C$ is symmetric, and complex conjugation is a field automorphism.

Clearly $A\alpha^{-n} \to 0$ as $n \to \infty$. The behavior of the other term $2\text{Re}[C\gamma^{-n}]$ can be understood with the help of the Euler formula

$$z = |z| \exp (i\zeta),$$

where the modulus $|z| = \sqrt{x^2 + y^2}$ of complex number $z = x + iy$ is the distance in the complex plane of $z$ from 0, and the argument $\zeta$ of $z$ is an angle in the complex plane from the positive real axis to the ray emanating from 0 to $z$. Complex multiplication is multiplicative in modulus and additive in argument. Because $|C\gamma^{-n}| \to \infty$ as $n \to \infty$, the desired conclusion holds if the ray emanating from 0 to $C\gamma^{-n}$ is near the negative real axis for some $n$ large enough that the second term $2\text{Re}[C\gamma^{-n}]$ of the sum is dominant.

Indeed, $2\text{Re}[C\gamma^{-n}]$ oscillates in sign. Because $|\alpha| > 1$, eventually $|A\alpha^{-n}| < 0.1$, say, while $2\text{Re}[C\gamma^{-n}] < -0.1$ infinitely often (cf. exercise II.1.11 of [3]).

□

Theorem 12. $5 \leq R$.

Proof. Fix some rational $r$ with $4.999 < r < 5$. Let $\Theta$ be the set of $\theta$ such that a $\theta$-quasicap exists. Clearly $1 \in \Theta$. Recall that $\gamma$ depends on $r$ and $\theta$. Though $r$ is fixed, $\theta$ is free, so $\gamma$ is a function of $\theta$. Let

$$F = \left\{ \frac{1}{1 - \gamma} - \theta \mid 1 \leq \theta \leq 2.13 \right\}.$$ 

Then $F$ is closed because it is the continuous image of a compact set. Propositions 7 and 8 imply $\inf F > 0$. Fix some rational $\delta$ with $0 < \delta < \inf F$ so that $(2.13 - 1)/\delta$ is an integer. Propositions 4, 5, and 6 imply $u_{n+1} - u_n \to -\infty$ for all $\theta \in [1, 2.13]$. Invoking Proposition 3 many (i.e. $1.13/\delta$) times yields $2.13 \in \Theta$. Propositions 3 and 9 imply $2.15 \in \Theta$ because $2.15 - 2.13 < 0.04$. Propositions 3, 10, and 11 imply $r - 2 \in \Theta$. That is, an $r$-cap exists. Proposition 2 implies $5 \leq R$. □

For some bad pairs $(r, \theta)$, no positive $\delta$ is small enough for Proposition 3. In the present section, we worked to avoid such sequences while getting as close as possible. The curve
\( \frac{1}{1-\gamma} = \theta \) with \( \delta = 0 \) in the \((r, \theta)\)-plane (see Figure 11) is an important part of the boundary of the bad region. Here:

\[
\begin{align*}
\theta &= \frac{1}{1-\gamma} \\
1 &= \theta(1-\gamma) \\
0 &= q(\gamma) \\
&= 1 - (r - \theta)\gamma + (r - 2\theta)\gamma^2 + \theta\gamma^3 \\
&= 1 - r(\gamma - \gamma^2) + \theta(\gamma - 2\gamma^2 + \gamma^3)
\end{align*}
\]

\[
\begin{align*}
r &= \frac{1 + \theta\gamma(1-\gamma)^2}{\gamma(1-\gamma)} \\
&= 1 + \theta/\gamma \\
&= 1 + \theta(1-\theta^{-1})^{-1}
\end{align*}
\]

and the minimum value of \( r \) on the curve occurs when \( dr/d\theta = 0, \theta = 2, \) and \( r = 5 \). The \( r \)-minimal bad sequence \( u_n \) is the one with \((r, \theta, \delta) = (5, 2, 0)\).

**Note.** If \((r, \theta, \delta) = (5, 2, 0)\), then \( u_n \) is the Fibonacci sequence.
5. No binary 5-cap exists

The constructions that yield this greatest known lower bound $5 \leq R$ are made according to helpful design choices. In the cap of Chrobak and Šlusarek, one pursues height while worrying only about cliques near cones. That cap also shows that each box can be supported on both sides of its cone. A heuristic arises that the supporting box on the side with more box weight above should cover the lower part of the gap, as both supporters must be supported themselves, and the supporter with greater box weight above, having a lower cone, should be compensated with the advantage of being lower.

A cap constructed according to these design choices, where each box has up to 2 supporters, one on each side of its cone, is binary. In order to be used in the construction of an interval graph, a binary cap can have only finitely many boxes. We will show that the copy-and-paste method of Section 3 yields the best possible binary caps. If someday these helpful design choices are shown to be optimal, this would imply $R = 5$.

An obvious scheme for addressing boxes of a binary cap $C$ is to name each box of $C$ by a binary word, as in Figure 12. Box $w$ may have a high child $wH$ and may have a low child $wL$. Indeed, we may describe the whole binary cap $C$ by a function $\kappa: \{H, L\}^* \to \mathbb{R}^+$ where $C$ has a box $w \in \{H, L\}^*$ if and only if $\kappa(w) > 0$. The word of length 0 is denoted $\lambda$. We assume $\kappa(w) > 0$ whenever $\kappa(wH) > 0$ or $\kappa(wL) > 0$. We consider only nonempty binary caps. In particular, $\kappa(\lambda) > 0$. Figure 12 depicts another important feature. Each box $wH$ (resp. $wL$) meets box $w$ at a vertical line called the seam of $wH$ (resp. $wL$).

Let $\kappa_w = \kappa(w)$. Let $\tau_w$ denote the depth\(^{21}\) of the top of the box corresponding to box $w$. The cumulative height of boxes above box $w$ is $\beta_w$, and $\pi_w$ (the “penalty”) is the difference of the cumulative height across the seam of $w$ and $\beta_w$. The Fibonacci sequence is the sequence $(f_0, f_1, \ldots) = (0, 1, \ldots)$ where $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

The following hold for boxes $\lambda, v, vL, vH$ in a binary $r$-cap.

---

\(^{21}\)This goes against the prior custom of recording levels as nonpositive. Depths usually are nonnegative.
\(0 = \beta_{\lambda} = \pi_{\lambda} = \tau_{\lambda};\)

\(\beta_{vL} = \beta_v + \pi_v;\)

\(\pi_{vL} = \kappa_v - \pi_v;\)

\(\kappa_{vL} \geq r(\beta_v + \kappa_v) - \tau_{vL} \geq 0;\)

\(\beta_{vH} = \beta_v;\)

\(\pi_{vH} = \kappa_v;\)

\(\kappa_{vH} \geq \tau_{vL} - \tau_v - \kappa_v \geq 0;\)

\(\tau_{vH} = \tau_v + \kappa_v.\)

**Definition.** Suppose \(w = uH^m\) is a box in a binary cap, and set \(w_i = uH^i\). Then \(w\) is \(m\)-hard if for all \(i\) with \(0 \leq i \leq m\) each of the following holds true:

- (H0) \(\pi_{w_i} \geq 0;\)
- (H1) \(\kappa_{w_i} \geq 2\pi_{w_i} - \pi_uf_i;\) and
- (H2) \(\tau_{w_i} \leq 5\beta_{w_i} + 2\pi_{w_i} + \pi uf_{i+1}.\)

A box is hard if it is \(m\)-hard for some \(m \in \{0, 1, \ldots\}\).

**Lemma 13.** If \(w = uH^m\) is \(m\)-hard then \(\kappa_w \geq \pi uf_{m+3}.\)

**Proof.** Argue by induction on \(m\). If \(m = 0\) then \(\kappa_w \geq 2\pi_w - \pi uf_0 = \pi uf_3.\) Else \(m > 0\). Then \(w' = uH^{m-1}\) is \((m-1)\)-hard. By (H1), (H-\(\pi\)) and induction, we have:

\[\kappa_w \geq 2\pi_w - \pi uf_m = 2\kappa_w - \pi uf_m \geq 2\pi uf_{m+2} - \pi uf_m = \pi uf_{m+2} + f_{m+1} = \pi uf_{m+3}.\]

\(\square\)

**Theorem 14.** There does not exist a binary 5-cap.

**Proof.** Suppose \(\kappa\) is a binary 5-cap. The top box \(\lambda\) is 0-hard as \(\kappa_\lambda > 0\) and \(\pi_\lambda = \tau_\lambda = 0.\) Since caps are finite, there is a hard box \(w\) with maximum word length. For a contradiction, we show that there exists \(w^+ \in \{wL, wH\}\) such that \(w^+\) is hard and \(\kappa_{w^+} > 0\) (so \(w^+\) is in \(\kappa\)).

Let \(w = uH^m\) be \(m\)-hard. If \(\tau_{wL} < 5\beta_w + 3\kappa_w + 2\pi_w\) then \(wL\) is 0-hard, since

\[\pi_{wL} = \kappa_w - \pi_w \geq 2\pi_w - \pi_w \geq 0\]  

((L-\(\pi\)), (H1), (H0))

and

\[\kappa_{wL} \geq 5\beta_w + 5\kappa_w - \tau_{wL}.\]

\[> 5\beta_w + 5\kappa_w - (5\beta_w + 3\kappa_w + 2\pi_w)\]  

(case)

\[= 2\kappa_w - 2\pi_w\]

\[= 2\pi_{wL} - \pi_{wL}f_0\]  

((L-\(\pi\)), Fibonacci)
and
\[ \tau_{wL} < 5\beta_w + 3\kappa_w + 2\pi_w + 3(\pi_w - \pi_w) \quad \text{(case)} \]
\[ = 5\beta_{wL} + 2\pi_{wL} + \pi_{wL}f_1. \quad \text{((L-\beta), (L-\pi), Fibonacci)} \]
Moreover, \( \kappa_{wL} > 2\pi_{wL} - \pi_{wL}f_0 = 2\pi_{wL} \geq 0 \), since in this case (H1) is strict.
Otherwise \( \tau_{wL} \geq 5\beta_w + 3\kappa_w + 2\pi_w \). Then \( wH = uH_{m+1} \) is \((m+1)-\text{hard, since}
\[ \pi_w = \kappa_{uH^m} \geq 0 \quad \text{((H-\pi))} \]
and
\[ \kappa_{wH} \geq \tau_{wL} - \tau_w - \kappa_w \quad \text{((H-\kappa))} \]
\[ \geq (5\beta_w + 3\kappa_w + 2\pi_w) - (5\beta_w + 2\pi_w + \pi_u f_{m+1}) - \kappa_w \quad \text{(case, (H2) for \( w \))} \]
\[ = 2\kappa_w - \pi_u f_{m+1} \]
\[ = 2\pi_{wH} - \pi_u f_{m+1} \quad \text{((H-\pi))} \]
and
\[ \tau_{wH} = \tau_w + \kappa_w \quad \text{((H-\tau))} \]
\[ \leq 5\beta_w + 2\pi_w + \pi_u f_{m+1} + \kappa_w \quad \text{((H2) for \( w \))} \]
\[ \leq 5\beta_{wH} + \kappa_w + (2\pi_w - \pi_u f_m) + \pi_u f_{m+2} \quad \text{((H-\beta), Fibonacci)} \]
\[ \leq 5\beta_{wH} + 2\pi_w + \pi_u f_{m+2} \quad \text{((H1) for \( w \))} \]
\[ \leq 5\beta_{wH} + 2\pi_{wH} + \pi_u f_{m+2}. \quad \text{((H-\pi))} \]
If \( \pi_u > 0 \) then \( \kappa_{wH} \geq \pi_u f_{m+4} > 0 \) by Lemma 13. If \( \pi_u = 0 \) then \( \kappa_{wH} \geq 2\pi_{wH} = 2\kappa_w > 0 \). □

References