ON THE CORRÁDI-HAJNAL THEOREM AND A QUESTION OF DIRAC

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Abstract. In 1963, Corrásdi and Hajnal proved that for all $k \geq 1$ and $n \geq 3k$, every graph $G$ on $n$ vertices with minimum degree $\delta(G) \geq 2k$ contains $k$ disjoint cycles. The bound $\delta(G) \geq 2k$ is sharp. Here we characterize those graphs with $\delta(G) \geq 2k - 1$ that contain $k$ disjoint cycles. This answers the simple-graph case of Dirac’s 1963 question on the characterization of $(2k - 1)$-connected graphs with no $k$ disjoint cycles.

Enomoto and Wang refined the Corrádi-Hajnal Theorem, proving the following Ore-type version: For all $k \geq 1$ and $n \geq 3k$, every graph $G$ on $n$ vertices contains $k$ disjoint cycles, provided that $d(x) + d(y) \geq 4k - 1$ for all distinct nonadjacent vertices $x, y$. We refine this further for $k \geq 3$ and $n \geq 3k + 1$: If $G$ is a graph on $n$ vertices such that $d(x) + d(y) \geq 4k - 3$ for all distinct nonadjacent vertices $x, y$, then $G$ has $k$ vertex-disjoint cycles if and only if the independence number $\alpha(G) \leq n - 2k$ and $G$ is not one of two small exceptions in the case $k = 3$. We also show how the case $k = 2$ follows from Lovász’ characterization of multigraphs with no two disjoint cycles.

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1. Introduction

For a graph $G = (V, E)$, let $|G| = |V|$, $|G| = |E|$, $\delta(G)$ be the minimum degree of $G$, and $\alpha(G)$ be the independence number of $G$. Let $\overline{G}$ denote the complement of $G$ and for disjoint graphs $G$ and $H$, let $G \lor H$ denote $G \cup H$ together with all edges from $V(G)$ to $V(H)$.

In 1963, Corrádi and Hajnal proved a conjecture of Erdős by showing the following:

Theorem 1.1 ([3]). Let $k \in \mathbb{Z}^+$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k$ contains $k$ disjoint cycles.

Clearly, hypothesis (i) in the theorem is sharp. Hypothesis (ii) also is sharp. Indeed, if a graph $G$ has $k$ disjoint cycles, then $\alpha(G) \leq |G| - 2k$, since every cycle contains at least two vertices of $G - I$ for any independent set $I$. Thus $H := K_{k+1} \cup K_{2k-1}$ satisfies (i) and has $\delta(H) = 2k - 1$, but does not have $k$ disjoint cycles, because $\alpha(H) = k + 1 > |H| - 2k$. There are several works refining Theorem 1.1. Dirac and Erdős [5] showed that if a graph $G$ has...
many more vertices of degree at least $2k$ than vertices of degree at most $2k-2$, then $G$ has $k$ disjoint cycles. Dirac [4] asked:

**Question 1.2.** Which $(2k-1)$-connected graphs do not have $k$ disjoint cycles?

He also resolved his question for $k = 2$ by describing all 3-connected multigraphs on at least 4 vertices in which every two cycles intersect. It turns out that the only simple 3-connected graphs with this property are wheels. Lovász [17] fully described all multigraphs in which every two cycles intersect.

The following result in this paper yields a full answer to Dirac’s question for simple graphs.

**Theorem 1.3.** Let $k \geq 2$. Every graph $G$ with (i) $|G| \geq 3k$ and (ii) $\delta(G) \geq 2k - 1$ contains $k$ disjoint cycles if and only if

(H3) $\alpha(G) \leq |G| - 2k$, and

(H4) if $k$ is odd and $|G| = 3k$, then $G \neq 2K_k \lor \overline{K}_k$ and if $k = 2$ then $G$ is not a wheel.

For fixed $k$, the conditions of Theorem 1.3 can be tested in polynomial time.

It is likely that Dirac intended his question to refer to multigraphs; indeed, his result for $k = 2$ is for multigraphs. On the other hand, the above-mentioned paper [5] by Dirac and Erdős is about simple graphs. In a forthcoming paper we will heavily use the results of this paper to obtain a characterization of $(2k-1)$-connected multigraphs that contain $k$ disjoint cycles, answering Question 1.2 in full.

Enomoto [6] and Wang [19] generalized the Corrádi-Hajnal Theorem in terms of the minimum Ore-degree $\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$:

**Theorem 1.4** ([6],[19]). Let $k \in \mathbb{Z}^+$. Every graph $G$ with (i) $|G| \geq 3k$ and

(E2) $\sigma_2(G) \geq 4k - 1$

contains $k$ disjoint cycles.

Again $H := \overline{K}_{k+1} \lor K_{2k-1}$ shows that hypothesis (E2) of Theorem 1.4 is sharp. What happens if we relax (E2) to (H2): $\sigma_2(G) \geq 4k - 3$, but again add hypothesis (H3)? Here are two interesting examples.

**Example 1.5.** Let $k = 3$ and $Y_1$ be the graph obtained by twice subdividing one of the edges $wz$ of $K_8$, i.e., replacing $wz$ by the path $wxyz$. Then $|Y_1| = 10 = 3k+1$, $\sigma_2(Y_1) = 9 = 4k-3$, and $\alpha(Y_1) = 2 \leq |Y_1| - 2k$. However, $Y_1$ does not contain $k = 3$ disjoint cycles, since each cycle would need to contain three vertices of the original $K_8$ (see Figure 1.1(a)).

![Figure 1.1](image-url)
Example 1.6. Let \( k = 3 \). Let \( Q \) be obtained from \( K_{4,4} \) by replacing a vertex \( v \) and its incident edges \( uv, vx, vy, vz \) by new vertices \( u', \) and edges \( uu', uv, ux, uy, uz \); so \( d(u) = 3 = d(u') \) and contracting \( uu' \) in \( Q \) yields \( K_{4,4} \). Now set \( Y_2 := K_1 \vee Q \). Then \( |Y_2| = 10 = 3k + 1, \sigma_2(Y_2) = 9 = 4k - 3, \) and \( \alpha(Y_2) = 4 \leq |Y_2| - 2k \). However, \( Y_2 \) does not contain \( k = 3 \) disjoint cycles, since each 3-cycle contains the only vertex of \( K_1 \) (see Figure 1.1(b)).

Our main result is:

Theorem 1.7. Let \( k \in \mathbb{Z}^+ \) with \( k \geq 3 \). Every graph \( G \) with

1. \( |G| \geq 3k + 1 \),
2. \( \sigma_2(G_k) \geq 4k - 3 \), and
3. \( \alpha(G) \leq |G| - 2k \)

contains \( k \) disjoint cycles, unless \( k = 3 \) and \( G \in \{Y_1, Y_2\} \). Furthermore, for fixed \( k \) there is a polynomial time algorithm that either produces \( k \) disjoint cycles or demonstrates that one of the hypotheses fails.

Theorem 1.7 is proved in Section 2. In Section 3 we discuss the case \( k = 2 \). In Section 4 we discuss connections to equitable colorings and derive Theorem 1.3 from Theorem 1.7 and known results.

Now we discuss examples demonstrating the sharpness of hypothesis (H2) that \( \sigma(G) \geq 4k - 3 \), and finally we review our notation.

Example 1.8. Let \( k \geq 3, Q = K_3 \) and \( G_k := \overline{K_{2k-2}} \vee (K_{2k-3} + Q) \). Then \( |G_k| = 4k - 2 \geq 3k + 1, \delta(G_k) = 2k - 2 \) and \( \alpha(G_k) = |G_k| - 2k \). If \( G_k \) contained \( k \) disjoint cycles, then at least \( 4k - |G_k| = 2 \) would be 3-cycles; this is impossible, since any 3-cycle in \( G_k \) contains an edge of \( Q \). This construction can be extended. Let \( k = r + t \), where \( k + 3 \leq 2r \leq 2k \), \( Q' = K_{2t} \), and put \( H = G_r \vee Q' \). Then \( |H| = 4r - 2 + 2t = 2k + 2r - 2 \geq 3k + 1, \delta(H) = 2r - 2 + 2t = 2k - 2 \) and \( \alpha(H) = 2r - 2 = |H| - 2k \). If \( H \) contained \( k \) disjoint cycles, then at least \( 4k - |H| = 2t + 2 \) would be 3-cycles; this is impossible, since any 3-cycle in \( H \) contains an edge of \( Q \) or a vertex of \( Q' \).

There are several special examples for small \( k \). The constructions of \( Y_1 \) and \( Y_2 \) can be extended to \( k = 4 \) at the cost of lowering \( \sigma_2 \) to \( 4k - 4 \). Below is another small family of special examples. The blow-up of \( G \) by \( H \) is denoted by \( G[H] \); that is, \( V(G[H]) = V(G) \times V(H) \) and \((x, y)(x', y') \in E(G[H]) \) if and only if \( xx' \in E(G) \), or \( x = x' \) and \( yy' \in E(H) \).

Example 1.9. For \( k = 4, G := C_5[K_3] \) satisfies \( |G| = 15 \geq 3k + 1, \delta(G) = 2k - 2 \) and \( \alpha(G) = 6 \leq |G| - 2k \). Since girth(G) = 4, G has at most \( |G|/4 < k \) disjoint cycles. This example can be extended to \( k = 5,6 \) as follows. Let \( I = K_{2k-8} \) and \( H = G \vee I \). Then \( |G| = 2k + 7 \geq 3k + 1, \delta(G) = 2k - 2 \) and \( \alpha(G) = 6 \leq |G| - 2k = 7 \). If \( H \) has \( k \) disjoint cycles then each of the at least \( k - (2k - 8) = 8 - k \) cycles that do not meet \( I \) use 4 vertices of \( G \), and the other cycles use at least 2 vertices of \( G \). So \( 15 = |G| \geq 2k + 2(8 - k) = 16, \) a contradiction.

Notation. A \emph{bud} is a vertex with degree 0 or 1. A vertex is \emph{high} if it has degree at least \( 2k - 1 \), and \emph{low} otherwise. For vertex subsets \( A, B \) of a graph \( G = (V, E) \), let

\[
\|A, B\| := \sum_{u \in A} |\{uv \in E(G) : v \in B\}|.
\]
Note $A$ and $B$ need not be disjoint. For example, $\|V,V\| = 2\|G\| = 2|E|$. We will abuse this notation to a certain extent. If $A$ is a subgraph of $G$, we write $\|A, B\|$ for $\|V(A), B\|$, and if $A$ is a set of disjoint subgraphs, we write $\|A, B\|$ for $\bigcup_{H \in A} V(H), B\|$. Similarly, for $u \in V(G)$, we write $\|u, B\|$ for $\{|u\}, B\|$. Formally, an edge $e = uv$ is the set $\{u, v\}$; we often write $\|e, A\|$ for $\{|u, v\}, A\|$.

If $T$ is a tree or a directed cycle and $u, v \in V(T)$ we write $uTv$ for the unique subpath of $T$ with endpoints $u$ and $v$. We also extend this: if $w \notin T$, but has exactly one neighbor $u \in T$, we write $wTv$ for $w(T + w + uw)v$. Finally, if $w$ has exactly two neighbors $u, v \in T$, we may write $wTvw$.

2. Proof of Theorem 1.7

Suppose $G = (V, E)$ is an edge-maximal counterexample to Theorem 1.7. That is, for some $k \geq 3$, (H1)–(H3) hold, and $G$ does not contain $k$ disjoint cycles, but adding any edge $e \in E(G)$ to $G$ results in a graph with $k$ disjoint cycles. The edge $e$ will be in precisely one of these cycles, so $G$ contains $k - 1$ disjoint cycles, and at least three additional vertices. Choose a set $C$ of disjoint cycles in $G$ so that:

(O1) $|C|$ is maximized;
(O2) subject to (O1), $\sum_{C \in C} |C|$ is minimized;
(O3) subject to (O1) and (O2), the length of a longest path $P$ in $R := G - \bigcup C$ is maximized;
(O4) subject to (O1), (O2), and (O3), $\|R\|$ is maximized.

Call such a $C$ an optimal set. We prove in Subsection 2.1 that $R$ is a path, and in Subsection 2.2 that $|R| = 3$. We develop the structure of $C$ in Subsection 2.3. Finally, in Subsection 2.4, these results are used to prove Theorem 1.7.

Our arguments will have the following form. We will make a series of claims about our optimal set $C$, and then show that if any part of a claim fails, then we could have improved $C$ by replacing a sequence $C_1, \ldots, C_t \in C$ of at most three cycles by another sequence of cycles $C'_1, \ldots, C'_{t'}$. Naturally, this modification may also change $R$ or $P$. We will express the contradiction by writing “$C'_1, \ldots, C'_{t'}, R', P'$ beats $C_1, \ldots, C_t, R, P$,” and may drop $R'$ and $R$ or $P'$ and $P$ if they are not involved in the optimality criteria.

This proof implies a polynomial time algorithm. We start by adding enough extra edges—at most $3k$—to obtain from $G$ a graph with a set $C$ of $k$ disjoint cycles. Then we remove the extra edges in $C$ one at a time. After removing an extra edge, we calculate a new collection $C'$. This is accomplished by checking the series of claims, each in polynomial time. If a claim fails, we calculate a better collection (again in polynomial time) and restart the check, or discover an independent set of size greater than $|G| - 2k$. As there can be at most $n^4$ improvements, corresponding to adjusting the four parameters (O1)–(O4), this process ends in polynomial time.

We now make some simple observations. Recall that $|C| = k - 1$ and $R$ is acyclic. By (O2) and our initial remarks, $|R| \geq 3$. Let $a_1$ and $a_2$ be the endpoints of $P$. (Possibly, $R$ is an independent set, and $a_1 = a_2$.)

Claim 2.1. For all $w, w' \in V(R)$ and $C \in C$, if $\|w, C\| \geq 2$ then $3 \leq |C| \leq 6 - \|w, C\|$. In particular, (a) $\|w, C\| \leq 3$, (b) if $\|w, C\| = 3$ then $|C| = 3$, and (c) if $|C| = 4$ then the two neighbors of $w$ in $C$ are nonadjacent.
Proof. Let \( \overrightarrow{C} \) be a cyclic orientation of \( C \). For distinct \( u, v \in N(w) \cap C \), the cycles \( wu\overrightarrow{C}vw \) and \( wu\overrightarrow{C}vw \) have length at least \( |C| \) by (O2). Thus \( 2||C|| \leq ||wu\overrightarrow{C}vw|| + ||wu\overrightarrow{C}vw|| = ||C|| + 4 \). So \( |C| \leq 4 \). Similarly, if \( ||w, C|| \geq 3 \) then \( 3||C|| \leq ||C|| + 6 \), and so \( |C| = 3 \). \( \Box \)

Claim 2.2. If \( xy \in E(R) \) and \( C \in C \) with \( |C| \geq 4 \) then \( N(x) \cap N(y) \cap C = \emptyset \).

2.1. \( R \) is a path. Suppose \( R \) is not a path. Let \( L \) be the set of buds in \( R \); then \( |L| \geq 3 \).

Claim 2.3. For all \( C \in \mathcal{C} \), distinct \( x, y, z \in V(C) \), \( i \in [2] \), and \( u \in V(R - P) \):

(a) \( \{ux, uy, a_i z\} \not\subseteq E \);
(b) \( ||\{u, a_i\}, C|| \leq 4 \);
(c) \( \{a_i x, a_i y, a_3-i z, zu\} \not\subseteq E \);
(d) \( ||\{a_1, a_2\}, C|| \geq 5 \) then \( ||u, C|| = 0 \);
(e) \( ||\{u, a_i\}, R|| \geq 1 \); in particular \( ||a_i, R|| = 1 \) and \( |P| \geq 2 \);
(f) \( 4 - ||u, R|| \leq ||\{u, a_i\}, C|| \) and \( ||\{u, a_i\}, D|| = 4 \) for at least \( |C| - ||u, R|| \) cycles \( D \in \mathcal{C} \).

Proof. (a) Else \( u\overrightarrow{x}(C - z)yu, Pa_i z \) beats \( C, P \) by (O3) (see Figure 2.1(a)).

(b) Else \( |C| = 3 \) by Claim 2.1. So there are distinct \( p, q, r \in V(C) \) with \( up, uq, a_ir \in E \), contradicting (a).

(c) Else \( a_i x(C - z)yu, (P - a_i)a_3-i z \)u beats \( C, P \) by (O3) (see Figure 2.1(b)).

(d) Suppose \( ||\{a_1, a_2\}, C|| \geq 5 \) and \( p \in N(u) \cap C \). By Claim 2.1, \( |C| = 3 \). Pick \( j \in [2] \) with \( pa_j \in E \), preferring \( ||a_j, C|| = 2 \). Then \( V(C) - p \subseteq N(a_{3-j}) \), contradicting (c).

(e) Since \( a_i \) is an end of the maximal path \( P, N(a_i) \cap R \subseteq P \); so \( a_i u \not\subseteq E \). By (b)

\[
4(k - 1) \geq ||\{u, a_i\}, V - R|| \geq 4k - 3 - ||\{u, a_i\}, R||.
\]

Thus \( ||\{u, a_i\}, R|| \geq 1 \). Hence \( G[R] \) has an edge, \( |P| \geq 2 \), and \( ||a_i, P|| = ||a_i, R|| = 1 \).

(f) By (2.1) and (e), \( ||\{u, a_i\}, V - R|| \geq 4|C| - ||u, R|| \). Using (b), this implies the second assertion, and \( ||\{u, a_i\}, C|| + 4(|C| - 1) \geq 4|C| - ||u, R|| \) implies the first assertion. \( \Box \)

![Figure 2.1. Claim 2.3](image)

Claim 2.4. \( |P| \geq 3 \). In particular, \( a_1a_2 \not\subseteq E(G) \).

Proof. Suppose \( |P| \leq 2 \). Then \( ||u, R|| \leq 1 \). As \( |L| \geq 3 \), there is a bud \( e \in L \setminus \{a_1, a_2\} \). By Claim 2.3(f), there exists \( C = z_1 \ldots z_1 \in \mathcal{C} \) such that \( ||\{c, a_1\}, C|| = 4 \) and \( ||\{c, a_2\}, C|| \geq 3 \).

If \( ||c, C|| = 3 \) then \( a_1c \) contradicts Claim 2.3(a). If \( ||c, C|| = 1 \) then \( ||\{a_1, a_2\}, C|| = 5 \), contradicting Claim 2.3(d). Therefore, we assume \( ||c, C|| = 2 = ||a_1, C|| \) and \( ||a_2, C|| \geq 1 \).

By Claim 2.3(a), \( N(a_1) \cup N(a_2) = N(c) \). So there exists \( z_i \in N(a_1) \cap N(a_2) \) and \( z_j \in N(c) - z_i \).

Then \( a_1a_2z_ia_1, cz_jz_j \pm 1 \) beats \( C, P \) by (O3). \( \Box \)
Claim 2.5. Let $c \in L - a_1 - a_2, C \in \mathcal{C}$, and $i \in [2]$.

(a) $\|a_1, C\| = 3$ if and only if $\|c, C\| = 0$, and if and only if $\|a_2, C\| = 3$.
(b) There is at most one cycle $D \in \mathcal{C}$ with $\|a_i, D\| = 3$.
(c) For every $C \in \mathcal{C}$, $\|a_i, C\| \geq 1$ and $\|c, C\| \leq 2$.
(d) If $\{a_i, C\}, \|C\| = 4$ then $\|a_i, C\| = 2 = \|c, C\|$.

Proof. (a) If $\|c, C\| = 0$ then by Claims 2.1 and 2.3(f), $\|a_i, C\| = 3$. If $\|a_i, C\| \geq 3$ then by Claim 2.3(b), $\|c, C\| \leq 1$. By Claim 2.3(f), $\|a_3 - i, C\| \geq 2$, and by Claim 2.3(d), $\|c, C\| = 0$.

(b) As $c \in L$, $\|c, R\| \leq 1$. Thus Claim 2.3(f) implies $\|c, D\| = 0$ for at most one cycle $D \in \mathcal{C}$.

(c) Suppose $\|c, C\| = 3$. By Claim 2.3(a), $\{a_1, a_2\}, C\| = 0$. By Claims 2.4 and 2.3(d):

$$4k - 3 \leq \|\{a_1, a_2\}, R \cup C \cup (V - R - C)\| \leq 2 + 0 + 4(k - 2) = 4k - 6,$$

a contradiction. So $\|c, C\| \leq 2$. Thus by Claim 2.3(f), $\|a_i, C\| \geq 1$.

(d) Now (d) follows from (a).

Claim 2.6. $R$ has no isolated vertices.

Proof. Suppose $c \in L$ is isolated. Fix $C \in \mathcal{C}$. By Claim 2.3(f), $\{c, a_1\}, C\| = 4$. By Claim 2.5(d), $\|a_1, C\| = 2 = \|c, C\|$; so $d(c) = 2(k - 1)$. By Claim 2.3(a), $N(a_1) \cap C = N(c) \cap C$. Let $w \in V(C) - N(c)$. Then $d(w) \geq 4k - 3 - d(c) = 2k - 1 = 2|\mathcal{C}| + 1$. So, either $\|w, R\| \geq 1$ or $|N(w) \cap D| = 3$ for some $D \in \mathcal{C}$. In the first case, $c(C - w)c$ beats $C$ by (O4). In the second case, by 2.5(c) there exists some $x \in N(a_1) \cap D$. So $c(C - w)c, w(D - x)w$ beats $C, D$ by (O3).

Claim 2.7. $L$ is an independent set.

Proof. Suppose $c_1c_2 \in E(L)$. By Claim 2.4, $c_1, c_2 \notin P$. By Claim 2.3(f) and using $k \geq 3$, there is $C \in \mathcal{C}$ with $\{c_1, c_2\}, C\| = 4$ and $\{a_1, c_2\}, C\|, \{a_2, c_1\}, C\| \geq 3$. By Claim 2.5(d), $\|a_1, C\| = 2 = \|c_1, C\|$; so $\|a_2, C\|, \|c_2, C\| \geq 1$. By Claim 2.3(a), $N(a_1) \cap C, N(a_2) \cap C \subseteq N(c_1) \cap C$. So there are distinct $x, y \in N(c_1) \cap C$ with $xa_1, xa_2, ya_1 \in E$. If $xc_2 \in E$ then $c_1c_2xc_1, ya_1Pa_2$ beats $C, P$ by (O3). Else $a_1Pa_2xa_1, c_1(C - x)c_2c_1$ beats $C, P$ by (O1).

Claim 2.8. If $|L| \geq 3$ then for some $D \in \mathcal{C}$, $\|l, C\| = 2$ for every $C \in \mathcal{C} - D$ and every $l \in L$.

Proof. Suppose some $D_1, D_2 \in \mathcal{C}$ and $l_1, l_2 \in L$ satisfy $D_1 \neq D_2$ and $\|l_1, D_1\| \neq 2 \neq \|l_2, D_2\|$. CASE 1: $l_j \notin \{a_1, a_2\}$ for some $j \in [2]$. Say $j = 1$. For $i \in [2]$: $\|a_i, l_1\|, D_1\| \neq 4$ by Claim 2.5(d); $\|a_i, l_1\|, D_2\| = 4$ by Claim 2.3(f); $\|a_i, D_2\| = 2$ by Claim 2.5(d). So $l_2 \notin \{a_1, a_2\}$. By Claim 2.7, $l_1l_2 \notin E(G)$. So Claim 2.5(c) yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k - 3) = 4k - 4.$$  

CASE 2: $\{l_1, l_2\} \subseteq \{a_1, a_2\}$. Let $c \in L - l_1 - l_2$. As above, $\|\{l_1, c\}, D_1\| \neq 4$, and so $\|c, D_2\| = 2 = \|l_1, D_2\|$. This implies $l_1 \neq l_2$. By Claim 2.5(a,c), $\|l_2, D_2\| = 1$. Thus $\|\{l_2, c\}, D_1\| = 4$; so $\|c, D_1\| = 2$, and $\|l_1, D_1\| = 1$. With Claim 2.4, this yields the contradiction:

$$4k - 3 \leq \|\{l_1, l_2\}, R \cup D_1 \cup D_2 \cup (V - R - D_1 - D_2)\| \leq 2 + 3 + 3 + 4(k - 3) = 4k - 4.$$  

Claim 2.9. $R$ is a subdivided star (possibly a path).
Proof. Suppose not. Then we claim $R$ has distinct leaves $c_1, d_1, c_2, d_2 \in L$ such that $c_1 Rd_1$ and $c_2 Rd_2$ are disjoint paths. Indeed, if $R$ is disconnected then each component has two distinct leaves by Claim 2.6. Else $R$ is a tree. As $R$ is not a subdivided star, it has distinct vertices $s_1$ and $s_2$ with degree at least three. Deleting the edges and interior vertices of $s_1 Rs_2$ yields disjoint trees containing all leaves of $R$. Let $T_i$ be the tree containing $c_i, d_i \in T_i$.

By Claim 2.8, using $k \geq 3$, there is a cycle $C \in C$ such that $\|l, C\| = 2$ for all $l \in L$. By Claim 2.3(a), $N(a_1) \cap C = N(l) \cap C = N(a_2) \cap C = \{w_1, w_3\}$ for $l \in L - a_1 - a_2$. Then replacing $C$ in $C$ with $w_1 c_1 Rd_1 w_1$ and $w_3 c_2 Rd_2 w_3$ yields $k$ disjoint cycles. □

Claim 2.10. $R$ is a path or a star.

![Figure 2.2. Claim 2.10](image)

Proof. By Claim 2.9, $R$ is a subdivided star. If $R$ is neither a path nor a star then there are vertices $r, p, d$ with $\|r, p\|, \|p, R\| \geq 3$, $\|p, R\| = 2$, $d \in L - a_1 - a_2$ and (say) $pa_1 \in E$. Then $a_2 Rd$ is disjoint from $pa_1$ (see Figure 2.2(a)). By Claim 2.5(c), $d(d) \leq 1 + 2(k - 1) = 2k - 1$. So

$$\|p, V - R\| \geq 4k - 3 - \|p, R\| = (2k - 1) = 2k - 4 \geq 2.$$ 

In each of the following cases, $R \cup C$ has two disjoint cycles, contradicting (O1).

**CASE 1:** $\|p, C\| = 3$ for some $C \in C$. Then $|C| = 3$. By Claim 2.5(a), if $\|d, C\| = 0$ then $\|a_1, C\| = 3 = \|a_2, C\|$. Then for $w \in C$, $w a_1 pw$ and $a_2 (C - w)a_2$ are disjoint cycles (see Figure 2.2(b)). Else by Claim 2.5(c), $\|d, C\|, \|a_2, C\| \in \{1, 2\}$. By Claim 2.3(f), $\{d, a_2\} = 3$, so there are $l_1, l_2 \in \{a_2, d\}$ with $\|l_1, C\| \geq 1$ and $\|l_2, C\| = 2$; say $w \in N(l_1) \cap C$. If $l_2 w \in E$ then $w_1 l_1 R l_2 w$ and $p(C - w)p$ are disjoint cycles (see Figure 2.2(c)); else $l_1 w R l_1$ and $l_2 (C - w)l_2$ are disjoint cycles (see Figure 2.2(d)).

**CASE 2:** There are distinct $C_1, C_2 \in C$ with $\|p, C_1\|, \|p, C_2\| \geq 1$. By Claim 2.8, for some $i \in [2]$ and all $c \in L$, $\|c, C_i\| = 2$. Let $w \in N(p) \cap C_i$. If $w a_1 \in E$ then $D := wpa_1 w$ is a cycle and $G[(C_i - w) \cup a_2 Rd]$ contains cycle disjoint from $D$. Else, if $w \in N(a_2) \cup N(d)$, say $w \in N(c)$, then $a_1 (C_i - w)a_1$ and $cwpRc$ are disjoint cycles. Else, by Claim 2.1 there exist vertices $u \in N(a_2) \cap N(d) \cap C_i$ and $v \in N(a_1) \cap C_i - u$. Then $ua_2 Rd$ and $a_1 v (C_i - u) wpa_1$ are disjoint cycles.

**CASE 3:** Otherwise. Then using (2.2), $\|p, V - R\| = 2 = \|p, C\|$ for some $C \in C$. In this case, $k = 3$ and $d(p) = 4$. By (H2), $d(a_2), d(d) \geq 5$. Say $C = \{C, D\}$. By Claim 2.3(b), $\|\{a_2, d\}, D\| \leq 4$. So

$$\|\{a_2, d\}, C\| = \|\{a_2, d\}, (V - R - D)\| \geq 10 - 2 - 4 = 4.$$ 

By Claim 2.5(c, d), $\|a_2, C\| = \|d, C\| = 2$ and $\|a_1, C\| \geq 1$. Say $w \in N(a_1) \cap C$. If $w p \in E$ then $dR d_{a_2} (C - w)d$ contains a cycle disjoint from $w a_1 w$. Else, by Claim 2.3(a) there exists
Lemma 2.11. R is a path.

Proof. Suppose R is not a path. Then it is a star with root r and at least three leaves, any of which can play the role of $a_i$ or a leaf in \(L-a_1-a_2\). Thus Claim 2.5(c) implies \(\|l, C\| \in \{1, 2\}\) for all \(l \in L\) and \(C \in C\). By Claim 2.8 there is \(D \in C\) such that for all \(l \in L\) and \(C \in C-D\), \(\|l, C\| = 2\). By Claim 2.3(f) there is \(l \in L\) such that for all \(c \in L-l\), \(\|c, D\| = 2\). Fix distinct leaves \(l', l'' \in L-l\).

Let \(Z = N(l') - R\) and \(A = V \setminus \left(Z \cup \{r\}\right)\). By the first paragraph, every \(C \in C\) satisfies \(\|Z \cap C\| = 2\). So \(|A| = |C| - 2k + 1\). For a contradiction, we show that \(A\) is independent.

Note \(A \cap R = L\), so by Claim 2.7, \(A \cap R\) is independent. By Claim 2.3(a),

\[
(2.3) \quad \text{for all } c \in L \text{ and for all } C \in C, N(c) \cap C \subseteq Z.
\]

So \(\|L,A\| = 0\). By Claim 2.1(c), for all \(C \in C\), \(C \cap A\) is independent. Suppose, for a contradiction, \(A\) is not independent. Then there exist distinct \(C_1, C_2 \in C\), \(v_1 \in A \cap C_1\), and \(v_2 \in A \cap C_2\) with \(v_1v_2 \in E\). Subject to this choose \(C_2\) with \(\|v_1,C_2\|\) maximum. Let \(Z \cap C_1 = \{x_1, x_2\}\) and \(Z \cap C_2 = \{y_1, y_2\}\).

CASE 1: \(\|v_1,C_2\| \geq 2\). Choose \(i \in [2]\) so that \(\|v_1,C_2-y_i\| \geq 2\). Then define \(C_1 := v_1(C_2-y_i)v_1, C_2 := l'x_1(C_1-v_1)x_2l',\) and \(P^* := y_ii''rl\) (see Figure 2.3(a)). By (2.3), \(P^*\) is a path and \(C^*_2\) is a cycle. So \(C_1, C^*_2, P^*\) beats \(C_1, C_2, P\) by (O3).

CASE 2: \(\|v_1,C_2\| \leq 1\). Then for all \(C \in C\), \(\|v_1,C\| \leq 2\) and \(\|v_1,C_2\| = 1\); so \(\|v_1,C\| = \|v_1,C_2 \cup (C-C_2)\| \leq 1 + 2(k-2) = 2k-3\). By (2.3) \(\|v_1,L\| = 0\) and \(d(l) \leq 2k-1\). So by (H2), \(\|v_1,r\| = \|v_1,R\| = (4k-3) - \|v_1,C\| - d(l) \leq (4k-3) - (2k-3) - (2k-1) = 1\), and \(v_1r \in E\). Let \(C_1 := l'x_1(C_1-v_1)x_2l', C_2 := l''y_1(C_2-v_2)y_2l'',\) and \(P^* := v_2v_1rl\) (see Figure 2.3(b)). Then \(C_1, C^*_2, P^*\) beats \(C_1, C_2, P\) by (O3). \(\square\)

![Figure 2.3](image-url)

2.2. \(|R| = 3\). By Lemma 2.11, \(R\) is a path, and by Claim 2.4, \(|R| \geq 3\). Next we prove \(|R| = 3\). First, we prove a claim that will also be useful in later sections.

Claim 2.12. Let \(C\) be a cycle, \(P = v_1v_2\ldots v_s\) be a path, and \(1 < i < s\). At most one of the following two statements holds.
we may assume

we can switch

when

We will repeatedly use Claim 2.12 to obtain a contradiction to (O1) by showing that $G$ holds then $\|v_i\|^2 < 2$.

Claim 2.13. If $C \in C$, $h \in \{2\}$ and $\|e_h, C\| \geq \|e_3-h, C\|$ then $\|C, F\| \leq 7$; if $\|C, F\| = 7$ then

$|C| = 3 \quad \|a_h, C\| = 2 \quad \|a_i, C\| = 3 \quad \|a_i^r, Ra_1, C\| = 2$ and $N(a_h) \cap C = N(e_{3-h}) \cap C$.

Proof. We will repeatedly use Claim 2.12 to obtain a contradiction to (O1) by showing that $G[C \cup R]$ contains two disjoint cycles. Suppose $\|C, F\| \geq 7$ and say $h = 1$. Then $\|e_1, C\| \geq 4$. So there is $x \in e_1$ with $\|x, C\| \geq 2$. Thus $|C| \leq 4$ by Claim 2.1, and if $|C| = 4$ then no vertex in $C$ has two adjacent neighbors in $F$. So (1) holds with $v_1 = a_1$ and $v_i = a_i'$, even when $|C| = 4$.

If $\|e_1, C\| = 4$, as is the case when $|C| = 4$, then $\|e_2, C\| \geq 3$. If $|C| = 4$ there is a cycle $D := yza_1'a_2y$ for some $y, z \in C$. As (a) holds, $G[a_1Ra_2 \cup C - y - z]$ contains another disjoint cycle. So $|C| = 3$. As (c) must fail with $v_i = a_i'$, (a) and (c) hold for $v_i = a_i'$ and $v_i = a_2$, a contradiction. So $\|e_1, C\| \geq 5$. If $\|a_1, C\| = 3$ then (a) and (c) hold with $v_i = a_1$ and $v_i = a_i'$. So $\|a_1, C\| = 2$, $\|a_i', C\| = 3$ and $\|a_i^rRa_2, C\| \geq 2$. If there is $b \in P - e_1$ and $c \in N(b) \cap V(C) \setminus N(a_1)$ then $G[a_1Ra_2 + c]$ and $G[a_1(C - c)a_1]$ both contain cycles. So for every $b \in R - e_1$, $N(b) \cap C \subseteq N(a_1)$. Then if $\|a_i^rRa_2, C\| \geq 3$, (c) holds for $v_i = a_1$ and $v_i = a_i'$, contradicting that (1) holds. So $\|a_i^rRa_2, C\| = \|e_1, C\| = 2$ and $N(a_1) = N(e_2)$.

Lemma 2.14. $|R| = 3$ and $m := \max\{|C| : C \in C\} = 4$.

Proof. Let $t = \{|C \in C : \|F, C\| \leq 6\}$ and $r = \{|C \in C : |C| \geq 5\}$. It suffices to show $r = 0$ and $|R| = 3$: then $m \leq 4$, and $|V(C)| = |G| - |R| \geq 3(k - 1) + 1$ implies some $C \in C$ has length 4. Choose $R$ so that:

(P1) $R$ has as few low vertices as possible, and subject to this
(P2) $R$ has a low end if possible.

Let $C \in C$. By Claim 2.13, $\|F, C\| \leq 7$. By Claim 2.1, if $|C| \geq 5$ then $\|a, C\| \leq 1$ for all $a \in F$; so $\|F, C\| \leq 4$. Thus $r \leq t$. Hence

(2.4) $2(4k - 3) \leq \|F, (V \setminus R) \cup R\| \leq (7k - 1) - t - 2r + 6 \leq 7k - t - 2r - 1$. 

So $5 - k \geq t + 2r \geq 3r \geq 0$. Since $k \geq 3$, this yields $3r \leq t + 2r \leq 2$, so $r = 0$ and $t \leq 2$, with $t = 2$ only if $k = 3$.

CASE 1: $k - t \geq 3$. That is, there exist distinct cycles $C_1, C_2 \in C$ with $\|F, C_i\| \geq 7$. In this case, $t \leq 1$: if $k = 3$ then $C = \{C_1, C_2\}$ and $t = 0$; if $k > 3$ then $t < 2$. For both $i \in [2]$, Claim 2.13 yields $\|F, C_i\| = 7$, $|C_i| = 3$, and there is $x_i \in V(C_i)$ with $\|x_i, R\| = 1$ and $\|y, R\| = 3$ for both $y \in V(C_i - x_i)$. Moreover, there is a unique index $j = \beta(i) \in [2]$ with $\|a_j', C_i\| = 3$. For $j \in [2]$, put $I_j := \{i \in [2] : \beta(i) = j\}$; that is, $I_j = \{i \in [2] : \|a_j', C_i\| = 3\}$. Then $V(C_i) - x_i = N(a_{\beta(i)}) \cap C_i = N(e_{\beta(i)} - x_i) \cap C_i$. As $x_i \in E$, one of $x_i, a_{\beta(i)}$ is high. As we can switch $x_i$ and $a_{\beta(i)}$ (by replacing $C_i$ with $a_{\beta(i)}(C_i - x_i)a_{\beta(i)}$ and $R$ with $R - a_{\beta(i)} + x_i$), we may assume $a_{\beta(i)}$ is high.
Suppose $I_j \neq \emptyset$ for both $j \in [2]$; say $\|a'_1, C_1\| = \|a'_2, C_2\| = 3$. Then for all $B \in C$ and $j \in [2]$, $a_j$ is high, and either $\|a_j, B\| \leq 2$ or $\|F, B\| \leq 6$. So since $t \leq 1$, $2k - 1 \leq d(a_j) = \|a_j, B \cup F\| + \|a_j, C - B\| \leq \|a_j, B\| + 1 + 2(k - 2) + t \leq 2k - 2 + \|a_j, B\|$.  

Thus $N(a_j) \cap B \neq \emptyset$ for all $B \in C$. Let $y_j \in N(a_{3-j}) \cap C_j$. Then using Claim 2.13, $y_j \in N(a_j)$, and $a'_1(C_1 - y_1)a'_1, a'_2(C_2 - y_2)a'_2, a_1a_2y_1a_1$ beats $C_1, C_2$ by (O1).

Otherwise, say $I_1 = \emptyset$. If $B \in C$ with $\|F, B\| \leq 6$ then $\|e_1, B\| + 2\|a_2, B\| \leq \|F, B\| + \|a_2, B\| \leq 9$. Thus, using Claim 2.13,

$$2(4k - 3) \leq d(a_1) + d(a'_1) + 2d(a_2) = 5 + \|e_1, C\| + 2\|a_2, C\| \leq 5 + 6(k - 1 - t) + 9t$$

$\Rightarrow 2k \leq 5 + 3t$.

Since $k - t \geq 3$ (by the case), we see $3(k - t) + (5 + 3t) \geq 3(3) + 2k$ and so $k \geq 4$. Since $t \leq 1$, in fact $k = 4$ and $t = 1$, and equality holds throughout: say $B$ is the unique cycle in $C$ with $\|F, B\| \leq 6$. Then $\|a_2, B\| = \|e_1, B\| = 3$. Using Claim 2.13, $d(a_1) + d(a'_1) = \|e_1, R\| + \|e_1, C - B\| + \|e_1, B\| = 3 + 4 + 3 = 10$, and $d(a_1), d(a_2) = (4k - 3) - d(a_2) = 13 - (1 + 4 + 3) = 5$, so $d(a_1) = d(a_2) = 5$. Note $a_1$ and $a_2$ share no neighbors: they share none in $R$ because $R$ is a path, they share none in $C - B$ by Claim 2.13, and they share no neighbor $b \in B$ lest $a_1a'_1ba_1$ and $a_2(B - b)a_2$ beat $B$ by (O1). Thus every vertex in $V - e_1$ is high.

Since $\|e_1, B\| = 3$, first suppose $\|a_1, B\| \geq 2$, say $B - b \subseteq N(a_1)$. Then $a_1(B - b)a_1, a'_1a'_2a_2b$ beat $B, R$ by (P1) (see Figure 2.4(a)). Now suppose $\|a'_1, B\| \geq 2$, this time with $B - b \subseteq N(a'_1)$. Since $d(a_1) = 5$ and $\|a_1, R \cup B\| \leq 2$, there exists $c \in C \in C - B$ with $a_1c \in E(G)$. Now $c \in N(a_2)$ by Claim 2.13, so $a'_1(B - b)a'_1, a'_2(C - c)a'_2,$ and $a_1ca_2b$ beat $B, C$, and $R$ by (P1) (see Figure 2.4(b)).

\begin{figure}[h]
\begin{center}
\begin{tikzpicture}
\node at (0,0) {$a_1$};
\node at (1,1) {$a'_1$};
\node at (2,0) {$a_2$};
\node at (1,-1) {$a'_2$};
\node at (0.5,-2) {$b$};
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0.5,-2) -- (0,0) -- (1,1) -- (0.5,-2);
\draw (1,-1) -- (0.5,-2);
\node at (4,0) {$a_1$};
\node at (5,1) {$a'_1$};
\node at (6,0) {$a_2$};
\node at (5,-1) {$a'_2$};
\node at (4.5,-2) {$c$};
\draw (4,0) -- (5,1) -- (6,0) -- (5,-1) -- (4,0);
\draw (5,-1) -- (4.5,-2);
\draw (5,1) -- (4.5,-2);
\draw (4.5,-2) -- (4,0);\end{tikzpicture}
\end{center}
\caption{Figure 2.4. Lemma 2.14, Case 1}
\end{figure}
for each $i$. Suppose $\|R, v_i\| \leq 2$ and hence $|R| + 3 \leq |R, C| \leq 2|C|$. It follows that $|R| = 5$, $|C| = 4$ and $\|R, w\| = 2$ for each $w \in C$. This in turn yields that $G[C \cup R]$ has no triangles and $\|v_i, C\| \leq 2$ for each $i \in [5]$. By Claim 2.13, $\|F, C\| \leq 6$, so $\|v_3, C\| = 2$. Thus we may assume that for some $w \in C$, $N(w) \cap R = \{v_1, v_3\}$. Then $\|e_2, C\| = \|e_2, C - w\| \leq 1$, lest there exist a cycle disjoint from $wv_1v_2v_3w$ in $G[C \cup R]$. So, $\|e_1, C\| \geq 8 - 1 - 2 = 5$, a contradiction to Claim 2.1(b). This yields $|R| \leq 4$.

**Claim 2.15.** Either $a_1$ or $a_2$ is low.

**Proof.** Suppose $a_1$ and $a_2$ are high. Then since $|R, V| \leq 19$, we may assume $a'_1$ is low. Suppose there is $c \in C$ with $ca_2 \in E$ and $\|a_1, C - c\| \geq 2$. If $a'_1c \in E$, then $R \cup C$ contains two disjoint cycles; so $a'_1c \not\in E$ and hence $c$ is high. Thus either $a_1(C - c)a_1$ is shorter than $C$ or the pair $a_1(C - c)a_1, ca_2a'_1$ beats $C, R$ by (P2). Thus if $ca_2 \in E$ then $\|a_1, C - c\| \leq 1$. As $a_2$ is high, $\|a_2, C\| \geq 1$ and hence $\|a_1, C\| = \|a_1, C \setminus N(a_2)\| + \|a_1, N(a_2)\| \leq 2$. Similarly, $\|a_1, D\| \leq 2$. Since $a_1$ is high, $\|a_1, C\| = \|a_1, D\| = 2$, and $d(a_1) = 5$. Hence

$$(2.5) \quad N(a_2) \cap C \subseteq N(a_1) \cap C \quad \text{and} \quad N(a_2) \cap D \subseteq N(a_1) \cap D.$$ 

As $a_2$ is high, $d(a_2) = 5$ and in (2.5) equalities hold. Also $d(a'_1) = 4 \leq d(a'_2)$.

If there are $c \in C$ and $i \in [2]$ with $ca_i, ca'_i \in E$ then by (O2), $|C| = 3$. Also $ca'_ia_ic, a'_3a_3a_3a_3, (C - c)$ beats $C, R$ by either (P1) or (P2). (Recall $N(a_1) \cap C = N(a_2) \cap C$ and neighbors of $a_2$ in $C$ are high.) So $N(a_i) \cap N(a'_i) = \emptyset$. Thus the set $N(a_i) - R = N(a_2) - R$ contains no low vertices. Also, if $\|a'_i, C\| \geq 1$ then $|C| = 3$: else $C$ has the form $c_1c_2c_3c_4c_1$, where $a_1c_1, a_1c_3 \in E$, and so $a_1a'_1c_2a_1, c_3c_4a_2a_2$ beats $C, R$ by either (P1) or (P2). Thus $|C| = 3$ and $a'_1c \in E$ for some $c \in V(C) - N(a_1)$. If $\|a'_i, C\| \geq 1$, we have disjoint cycles $ca'_ia'_2c, a_1(C - c)a_1$ and $D$. Then $\|a'_1, C\| = 0$, so $d(a'_1) \leq 2 + |D \setminus N(a_1)| \leq 4$. Now $a'_1$ and $a'_2$ are symmetric, and we have proved that $\|a'_1, C\| + \|a'_2, C\| \leq 1$. Similarly, $\|a'_1, D\| + \|a'_2, D\| \leq 1$, a contradiction to $d(a'_1), d(a'_2) \geq 4$.

By Claim 2.15, we can choose notation so that $a_1$ is low.

**Claim 2.16.** If $a'_1$ is low then each $v \in V \setminus e_1$ is high.

**Proof.** Suppose $v \in V - e_1$ is low. Since $a_1$ is low, all vertices in $R - e_1$ are high, so $v \in C$ for some $C \in C$. Then $C' := ve_1v$ is a cycle and so by (O2), $|C'| = 3$. Since $a_2$ is high, $\|a_2, C\| \geq 1$. As $v$ is low, $va_2 \not\in E$. Since $a'_1$ is low, it is adjacent to the low vertex $v$, and $\|a'_1, C - v\| \leq 1$. So $C', a_2a_2(C - v)$ beats $C, R$ by (P1).

**Claim 2.17.** If $|C| = 3$ and $\|e_1, C\|, \|e_2, C\| \geq 3$, then either

(a) $\|c, e_1\| = 1 = \|c, e_2\|$ for all $c \in V(C)$ or

(b) $a'_1$ is high and there is $c \in V(C)$ with $\|c, R\| = 4$ and $C - c$ has a low vertex.

**Proof.** If (a) fails then $\|c, e_1\| = 2$ for some $i \in [2]$ and $c \in C$. If $\|e_3 - i, C - c\| \geq 2$ then there is a cycle $C' \subseteq C \cup e_3 - i - c$, and $R \cup C$ contains disjoint cycles $ce_iC$ and $C'$. Else,

$$\|c, R\| = \|c, e_i\| + (\|C, e_3 - i\| - \|C - c, e_3 - i\|) \geq 2 + (3 - 1) = 4 = |R|.$$ 

If $C - c$ has no low vertices then $ce_1C, e_2(C - c)$ beats $C, R$ by (P1). So $C - c$ contains a low $c'$. If $a'_1$ is low then $c'a'_1a_1c'$ and $ca_2a'_2c$ are disjoint cycles. So (b) holds. 

\[\square\]
CASE 2.1: \(|D| = 4\). By (O2), \(G[R ∪ D]\) does not contain a 3-cycle. So \(5 ≤ d(a_2) ≤ 3 + \|a_2', C\| ≤ 6\). Thus \(d(a_1), d(a_1') ≥ 3\).

Suppose \(\|e_1, D\| ≥ 3\). Pick \(v ∈ N(a_1) ∩ D\) with minimum degree, and \(v' ∈ N(a_1') ∩ D\). Since \(N(a_1) ∩ D\) and \(N(a_1') ∩ D\) are nonempty, disjoint and independent, \(vv' ∈ E\). Say \(D = vvv'v'v\). As \(D = K_2\) and low vertices are adjacent, \(D' := a_1a_1'v'va_1\) is a 4-cycle and \(v\) is the only possible low vertex in \(D\). Note \(a_1w ∉ E\): else \(a_1ww'va_1, v'a_1a_2a_2\) beats \(D, R\) by (P1). As \(\|e_1, D\| ≥ 3, a_1'v' ∈ E\). Also note \(\|e_2, wvw'\| = 0\): else \(G[a_2, a_2, w, w']\) contains a 4-path \(R'\), and \(D', R'\) beats \(D, R\) by (P1). Similarly, replacing \(D'\) by \(D'' := a_1a_1'w'va_1\) yields \(\|e_2, v'\| = 0\). So \(\|e_1 ∪ e_2, D\| ≤ 3 + 1 = 4\), a contradiction. Thus (2.6)

\[
\|e_1, D\| ≤ 2 \quad \text{and so} \quad \|R, D\| ≤ 6.
\]

Suppose \(d(a_1') = 3\). Then \(\|a_1', D\| ≤ 1\). So there is \(uv ∈ E(D)\) with \(\|a_1', uv\| = 0\). Thus \(d(u), d(v), d(a_2) ≥ 6\), and \(\|a_2, C\| = 3\). So \(|C| = 3\), \(|G| = 11\), and there is \(w ∈ N(u) ∩ N(v)\). If \(w ∈ C\) put \(C' = a_2(C - w) a_2\); else \(C' = C\). In both cases, \(|C'| = |C|\) and \(|wvwv| = 3 < |D|\), so \(C', wvwv\) beats \(C, D\) by (O2). Thus \(d(a_1') ≥ 4\). If \(d(a_1) = 3\) then \(d(a_2), d(a_2') ≥ 9 - 3 = 6\), and \(\|a_2, C\| ≥ 3\). By (2.6),

\[
\|R, C\| ≥ 3 + 4 + 6 + 6 - \|R, R\| - \|R, D\| ≥ 19 - 6 - 6 = 7,
\]

contradicting Claim 2.13. So \(d(a_1) = 4 ≤ d(a_1')\) and by (2.6), \(\|e_1, C\| ≥ 3\). Thus (2.6) fails for \(C\) in place of \(D\): so \(|C| = 3\). As \(\|a_2, C\| ≥ 2\) and \(\|a_2', C\| ≥ 1\), Claim 2.17 implies either (a) or (b) of Claim 2.17 holds. If (a) holds then (a) and (d) of Claim 2.12 both hold, and so \(G[C ∪ R]\) has two disjoint cycles. Else, Claim 2.17 gives \(a_1'\) is high and there is \(c ∈ C\) with \(\|c, R\| = 4\). As \(a_1'\) is high, \(\|R, C\| ≥ 7\). So \(\|c, R\| = 4\) contradicts Lemma 2.13.

CASE 2.2: \(|C| = |D| = 3\) and \(\|R, V\| = 18\). Then \(d(a_1) + d(a_2') = 9 = d(a_1') + d(a_2), a_1\) and \(a_1'\) are low, and by Claim 2.16 all other vertices are high. Moreover, \(d(a_1') ≤ d(a_1)\), since

\[
18 = \|R, V\| = d(a_1') - d(a_1) + 2d(a_1) + d(a_2) + d(a_2') ≥ d(a_1') - d(a_1) + 9 + 9.
\]

Suppose \(d(a_1') = 2\). Then \(d(v) ≥ 7\) for all \(v ∈ V - a_1a_1' a_2'.\) In particular, \(C ∪ D ⊆ N(a_2').\) If \(d(a_1) = 2\) then \(d(a_2') ≥ 7\), and \(G = Y_1\). Else \(\|a_1, C ∪ D\| ≥ 2\). If there is \(c ∈ C\) with \(V(C) - c ⊆ N(a_1)\), then \(a_1(C - c) a_1, a_1'a_1' a_2 c\) beats \(C, R\) by (P1). Else \(d(a_1) = 3\), \(d(a_2') = 6\), and there are \(c ∈ C\) and \(d ∈ D\) with \(c, d ∈ N(a_1)\). If \(ca_2' ∈ E\) then \(C ∪ R\) contains disjoint cycles \(a_1'ca_2'a_1'a_1\) and \(a_2'(C - c) a_2\), so assume not. Similarly, assume \(a_2' a_2' ∉ E\). Since \(d(d) ≥ 7\) and \(a_1', a_2' ∉ N(d)\), \(cd ∈ E(G)\). Then there are three disjoint cycles \(a_2'(C - c) a_2', a_2(D - d) a_2\), and \(a_1' c a_2\). So \(d(a_1') ≥ 3\).

Suppose \(d(a_1') = 3\). Say \(a_1'v ∈ E\) for some \(v ∈ D\). As \(d(a_3) ≥ 6\), \(\|a_2, D\| ≥ 2\). So \(e_2 + D - v\) contains a 4-path \(R'\). Thus \(a_1'v ∉ E\): else \(ve_1v, R'\) beats \(D, R\) by (P1). Also \(\|a_1, D - v\| ≤ 1\): else \(a_1(D - v) a_1, va_2'a_2 a_2\) beats \(D, R\) by (P1). So \(\|a_1, D\| ≤ 1\).

Suppose \(\|a_1, C\| ≥ 2\). Pick \(c ∈ C\) with \(C - c ⊆ N(a_1)\). Then (*) \(a_2c ∉ E\): else \(a_1(C - c)a_1, a_1'a_2'a_2c\) beats \(C, R\) by (P1). So \(\|a_2, C\| = 2\) and \(\|a_2, D\| = 3\). Also \(a_1c ∉ E\): else picking a different \(c\) violates (*). As \(a_1c ∉ E\), \(\|c, D\| = 3\) and \(a_2c ∈ E(G)\). So \(a_1(C - c) a_1, a_2(D - v) a_2\) and \(ca_1'a_2c\) are disjoint cycles. Otherwise, \(\|a_1, C\| ≤ 1\) and \(d(a_1) ≤ 3\). Then \(d(a_1) = 3\) since \(d(a_1) ≥ d(a_1')\).

Now \(d(a_2') = 6\). Say \(D = vbb'v\) and \(a_1b ∈ E\). As \(b'a_1' ∉ E, d(b') ≥ 9 - 3 = 6\). Since \(\|e_2, V\| = 12\), \(a_2\) and \(a_2'\) have three common neighbors. If one is \(b'\) then \(D' := a_1a_1'vbvb_1, b'e_2b'\), and \(C\) are disjoint cycles; else \(\|b', C\| = 3\) and there is \(c' ∈ C\) with \(\|c', e_2\| = 2\). Then \(D', c'e_2c'\) and \(b'(C - c')b'\) are disjoint cycles. So \(d(a_1') = 4\).
Since \(a_1\) is low and \(d(a_1) \geq d(a'_1)\), \(d(a_1) = d(a'_1) = 4\) and \(||\{a_1, a'_1\}, C \cup D|| = 5\), so we may assume \(||e_1, C|| \geq 3\). If \(||e_2, C|| \geq 3\), then because \(a'_1\) is low, Claim 2.17(a) holds. So \(V(C) \subseteq N(e_1)\) and there is \(x \in e_1 = xy\) with \(||x, C|| \geq 2\). First suppose \(||x, C|| = 3\). As \(x\) is low, \(x = a_1\). Pick \(c \in N(a_2) \cap C\), which exists because \(||a_2, C \cup D|| \geq 4\). Then \(a_1(C - c)a_1, a'_1a_2a_2c\) beats \(C, R\) by (P1). Now suppose \(||x, C|| = 2\). Let \(c \in C \setminus N(x)\). Then \(x(C - c)x, yce_2\) beats \(C, R\) by (P1).

CASE 2.3: \(|C| = |D| = 3\) and \(||R, V|| = 19\). Say \(||C, R|| = 7\) and \(||D, R|| = 6\).

CASE 2.3.1: \(a'_1\) is low. Then \(||a'_1, C \cup D|| \leq 4 - ||a'_1, R|| = 2\), so by Claim 2.13 \(||e_2, C|| = 5\) with \(||e_2, C|| = 2\). Then \(5 \leq d(a_2) \leq 6\).

If \(d(a_2) = 5\) then \(d(a_1) = d(a'_1) = 4\) and \(d(a'_2) = 6\). So \(||a_2, D|| = 2\) and \(||a'_2, D|| = 1\). Say \(D = b_1b_2b_3b_1\), where \(a_2b_2, a_2b_3 \in E\). As \(a'_1\) is low, (a) of Claim 2.17 holds. So \(||b_1, a_1a'_2|| = 2\), and there is a cycle \(D' \subseteq G[b_1a_1a'_1a'_2]\). Then \(a_2D - b_1a_2\) and \(D'\) are disjoint.

If \(d(a_2) = 6\) then \(||a_2, D|| = 3\). Let \(c_1 \in C - N(a_2)\). By Claim 2.13, \(||c_1, R|| = 1\), so \(c_1\) is high, and \(||c_1, D|| \geq 2\). If \(||a'_2, D|| \geq 1\), then (a) and (d) hold in Claim 2.12 for \(v_1 = a_2\) and \(v_1 = a'_2\), so \(G[D \cup c_1a'_2a_2]\) has two disjoint cycles, and \(c_2e_1c_3c_2\) contains a third. So assume \(||a'_2, D|| = 0\), and so \(||a'_2|| = 5\). Thus \(d(a_1) = d(a'_1) = 4\). Again, \(||e_1, D|| = 3 = ||a_2, D||\). So there are \(x \in e_1\) and \(b \in V(D)\) with \(D - b \subseteq N(x)\). As \(a'_1\) is low and has two neighbors in \(R\), if \(||x, D|| = 3\) then \(x = a_1\). Anyway, using Claim 2.17, \(G[R + b - x]\) contains a 4-path \(R'\), and \(x(D - b)x\) beats \(D, R\) by (P1).

CASE 2.3.2: \(a'_1\) is high. Since \(19 = ||R, V|| \geq d(a_1) + d(a'_1) + 2(9 - d(a_1)) \geq 23 - d(a_1)\), we get \(d(a_1) = 4\) and \(d(a'_1) = d(a'_2) = d(a_2) = 5\). Choose notation so that \(C = c_1c_2c_3c_1, D = b_1b_2b_3b_1\), and \(||c_1, R|| = 1\). By Claim 2.13, there is \(i \in [2]\) with \(||a_i, C|| = 2\), \(||a'_i, C|| = 3\), and \(a_i, c_1 \notin E\). If \(i = 1\) then every low vertex is in \(N(a_1) - a'_1 \subseteq D \cup C'\), where \(C' = a'_1c_3a_1\). So \(C', c_1a'_1a'_2a_2\) beats \(C, R\) by (P1). Thus let \(i = 2\). Now \(||a_2, C|| = 2 = ||a_2, D||\).

Say \(a_2b_2, a_2b_3 \in E\). Also \(||a_2, D|| = 0\) and \(||a_1, D|| = 4\). So \(||b_2, e_1|| = 2\) for some \(j \in [3]\). If \(j = 1\) then \(b_1e_1b_1\) and \(a_2b_2b_3a_2\) are disjoint cycles. Else, say \(j = 2\). By inspection, all low vertices are contained in \(\{a_1, b_1, b_3\}\). If \(b_1, b_3\) are high then \(b_2e_1b_2, b_1b_2e_2\) beats \(D, R\) by (P1). Else there is a 3-cycle \(D' \subseteq G[D + a_1]\) that contains every low vertex of \(G\). Pick \(D'\) with \(b_1 \in D'\) if possible. If \(b_2 \notin D'\) then \(D'\) and \(b_2a'_1a'_2a_2b_2\) are disjoint cycles. If \(b_3 \notin D'\) then \(D'\), \(b_3a_2a'_2a'_1\) beats \(D, R\) by (P1). Else \(b_1 \notin D', a_1b_1 \notin E\), and \(b_1\) is high. If \(b_1a'_1 \in E\) then \(D', b_1a'_1a'_2a_2\) beats \(D, R\) by (P1). Else, \(||b_1, C|| = 3\). So \(D', b_1c_1c_2b_1, a_3c_2c_3\) are disjoint cycles.

\(\square\)

2.3. Key Lemma. Now \(||R|| = 3\); say \(R = a_1a'_2a_2\). By Lemma 2.14 the maximum length of a cycle in \(C\) is 4. Fix \(C = w_1 \ldots w_4w_1 \in C\).

Lemma 2.18. If \(D \in C\) with \(||R, D|| \geq 7\) then \(||D|| = 3\), \(||R, D|| = 7\) and \(G[R \cup D] = K_6 - K_3\).

Proof. Since \(||R, D|| \geq 7\), there exists \(a \in R\) with \(||a, D|| \geq 3\). So \(||D|| = 3\) by Claim 2.1. If \(||a_i, D|| = 3\) for any \(i \in [2]\), then (a) and (c) in Claim 2.12 hold, violating (O1). Then \(||a_i, D|| = ||a_2, D|| = 2\) and \(||a'_2, D|| = 3\). If \(G[R \cup D] \neq K_6 - K_3\) then \(N(a_1) \cap D \neq N(a_2) \cap D\). Then there is \(w \in N(a_1) \cap D\) with \(||a_2, D - w|| = 2\). Then \(w_1a'_1w\) and \(a_2(D - w)a_2\) are disjoint cycles. \(\square\)

Lemma 2.19. Let \(D \in C\) with \(D = z_1 \ldots z_1z_1\). If \(||C, D|| \geq 8\) then \(||C, D|| = 8\) and

\[ W := G[C \cup D] \in \{K_{4,4}, K_1 \vee K_{3,3}, \overline{K}_3 \vee (K_1 + K_3)\}. \]

Then $C' := C - C - D + T + C'$ is at least as good as $C$. So by Lemma 2.14, $|C'| \leq 4$. Thus $C'$ beats $C$ by (O2).

CASE 1: $\Delta(W) = 6$. By symmetry, assume $d_W(w_4) = 6$. Then $\|\{z_i, z_{i+1}\}, C - w_4\| \geq 2$ for some $i \in \{1, 3\}$. So (*) holds with $T = w_1z_{i-1}z_{i+1}w_4$. 

CASE 2: $\Delta(W) = 5$. Say $z_1, z_2, z_3 \in N(w_1)$. Then $\|\{z_i, z_4\}, C - w_1\| \geq 2$ for some $i \in \{1, 3\}$. So (*) holds with $T = w_1z_{i-1}z_{i+1}w_1$.

CASE 3: $\Delta(W) = 4$. Then $W$ is regular. If $W$ has a triangle then (*) holds. Else, say $w_1z_1, w_1z_3 \in E$. Then $z_1, z_3 \not\in N(w_2) \cup N(w_4)$, so $z_2, z_4 \in N(w_2) \cup N(w_4)$, and $z_1, z_3 \in N(w_3)$.

Now, suppose $|D| = 3$.

CASE 1: $d_W(z_h) = 6$ for some $h \in \{3\}$. Say $h = 3$. If $w_i, w_{i+1} \in N(z_j)$ for some $i \in \{4\}$ and $j \in \{2\}$, then $z_3w_{i+2}w_{i+3}z_3, z_jw_{i+1}z_j$ beats $C, D$ by (O2). Else for all $j \in \{2\}, \|z_j, C\| = 2$, and the neighbors of $z_j$ in $C$ are nonadjacent. If $w_i \in N(z_1) \cup N(z_2) \cup C$, then $z_3w_{i+1}w_{i+2}z_3, z_1z_2w_{i+1}$ are preferable to $C, D$ by (O2). Wence $W = K_1 \lor K_{3,3}$.

CASE 2: $d_W(z_h) \leq 5$ for every $h \in \{3\}$. Say $d(z_1) = 5 = d(z_2), d(z_3) = 4$, and $w_1, w_2, w_3 \in N(z_1)$. If $N(z_1) \cap C \neq N(z_2) \cap C$ then $W - z_3$ contains two disjoint cycles, preferable to $C, D$ by (O2); if $w_i \in N(z_3)$ for some $i \in \{1, 3\}$ then $W - w_4$ contains two disjoint cycles. So $N(z_3) = \{w_2, w_4\}$, and $W = K_3 \lor (K_1 + K_3)$, where $V(K_1) = \{w_4\}, w_2z_1w_2w_3 = K_3$, and $V(K_3) = \{w_1, w_3, z_3\}$.

Claim 2.20. For $D \in C$, if $\|\{w_1, w_3\}, D\| \geq 5$ then $\|C, D\| \leq 6$. If also $|D| = 3$ then $\|\{w_2, w_4\}, D\| = 0$.

Proof. Assume not. Let $D = z_1 \ldots z_t z_1$. Then $\|\{w_1, w_3\}, D\| \geq 5$ and $\|C, D\| \geq 7$. Say $\|w_1, D\| \geq \|w_3, D\|$, $\{z_1, z_2, z_3\} \subseteq N(w_1)$, and $z_1 \in N(w_3)$.

Suppose $\|w_1, D\| = 4$. Then $|D| = 4$. If $\|z_3, C\| \geq 3$ for some $h \in \{4\}$ then there is a cycle $B \subseteq G[w_2, w_3, w_4, z_h]$; so $B, w_1z_{h+1}z_{h+2}w_1$ beats $C, D$ by (O2). Else there are $j \in \{l-1, l+1\}$ and $i \in \{2, 3, 4\}$ with $z_2w_i \in E$. Then $z_i z_j w_i w_3 z_i, w_1(D - z_i - z_j) w_1$ beats $C, D$ by (O2), where $w_3w_3 = w_3$ if $i = 3$.

Else, $\|w_1, D\| = 3$. By assumption, there is $i \in \{2, 4\}$ with $\|w_i, D\| \geq 1$. If $|D| = 3$, applying Claim 2.12 with $P := w_1w_3w_3$ and cycle $D$ yields two disjoint cycles in $(D \cup C) - w_{6-i}$, contradicting (O2). So suppose $|D| = 4$. Because $w_1z_1z_2w_1$ and $w_2z_2z_3w_3$ are triangles, there do not exist cycles in $G[\{w_2, w_3, z_3, z_1\}]$ or $G[\{w_3, z_1, z_3\}]$ by (O2). Then $\|\{w_i, w_3\}, \{z_3, z_1\}\|, \|\{w_i, w_3\}, \{z_1, z_4\}\| \leq 1$. Since $\|\{w_i, w_3\}, D\| \geq 3$, one has a neighbor in $z_2$. If both are adjacent to $z_2$, then $w_iw_3z_2w_i, w_1z_1z_2z_3w_1$ beat $C, D$ by (O2). Then $\|\{w_i, w_3\}, z_2\| = 1 = \|\{w_i, w_3\}, z_1\| = \|\{w_i, w_3\}, z_3\|$. Let $z_m$ be the neighbor of $w_i$. Then $w_1w_1w_3w_1, w_3(D - z_m)w_1$ beat $C, D$ by (O2).

Suppose $|D| = 3$ and $\|\{w_1, w_3\}, D\| \geq 5$. If $\|\{w_2, w_4\}, D\| \geq 1$, then $C \cup D$ contains two triangles, and these are preferable to $C, D$ by (O2). 

For $v \in N(C)$, set $\text{type}(v) = i \in \{2\}$ if $N(v) \cap C \subseteq \{w_1, w_{i+2}\}$. Call $v$ light if $\|v, C\| = 1$; else $v$ is heavy. For $D = z_1 \ldots z_3z_1 \in C$, put $H := H(D) := G[R \cup D]$.

Claim 2.21. If $\|\{a_1, a_2\}, D\| \geq 5$ then there exists $i \in \{2\}$ such that

(a) $\|C, H\| \leq 12$ and $\|\{w_i, w_{i+2}\}, H\| \leq 4$;
(b) $\|C, H\| = 12$;
(c) $N(w_i) \cap H = N(w_{i+2}) \cap H = \{a_1, a_2\}$ and $N(w_{3-i}) \cap H = N(w_{5-i}) \cap H = V(D) \cup \{a'\}$.
Proof. By Claim 2.1, \(|D| = 3\). Choose notation so that \(||a_1, D|| = 3\) and \(z_2, z_3 \in N(a_2)\).

(a) Using that \(\{w_1, w_3\}\) and \(\{w_2, w_4\}\) are independent and Lemma 2.19:

\[
\|C, H\| = \|C, V - (V - H)\| \geq 2(4k - 3) - 8(k - 2) = 10.
\]

Let \(v \in V(H)\). As \(K_4 \subseteq H\), \(H - v\) contains a 3-cycle. If \(C + v\) contains another 3-cycle then these 3-cycles beat \(C, D\) by (O2). So type(\(v\)) is defined for all \(v \in N(C) \cap H\), and \(\|C, H\| \leq 12\). If only five vertices of \(H\) have neighbors in \(C\) then there is \(i \in [2]\) such that at most two vertices in \(H\) have type \(i\). So \(\|\{w_i, w_{i+2}\}, H\| \leq 4\). Else every vertex in \(H\) has a neighbor in \(C\). By (2.7), \(H\) has at least four heavy vertices.

Let \(H'\) be the spanning subgraph of \(H\) with \(xy \in E(H')\) iff \(xy \in E(H)\) and \(H - \{x, y\}\) contains a 3-cycle. If \(xy \in E(H')\) then \(N(x) \cap N(y) \cap C = \emptyset\) by (O2). So if \(x\) and \(y\) have the same type they are both light. By inspection, \(H' \supseteq z_1a_1d_3z_2 + a_2z_3\).

Let type(\(a_2\)) = \(i\). If \(a_2\) is heavy then its neighbors \(a', z_2, z_3\) have type \(3 - i\). Either \(z_1, a_1\) are both light or they have different types. Anyway, \(\|\{w_i, w_{i+2}\}, H\| \leq 4\). Else \(a_2\) is light. Then because there are at least four heavy vertices in \(H\), at least one of \(z_1, a_1\) is heavy and so they have different types. Also any type-\(i\) vertex in \(a', z_2, z_3\) is light, but at most one vertex of \(a, z_2, z_3\) is light because there are at most two light vertices in \(H\). So \(\|\{w_i, w_{i+2}\}, H\| \leq 4\).

(b) By (a), there is \(i\) with \(\|\{w_i, w_{i+2}\}, H\| \leq 4\); thus

\[
\|\{w_i, w_{i+2}\}, V - H\| \geq (4k - 3) - 4 = 4(k - 2) + 1.
\]

So \(\|\{w_i, w_{i+2}\}, D'\| \geq 5\) for some \(D' \in C - C - D\). By (a), Claim 2.20, and Lemma 2.19,

\[
12 \geq \|C, H\| = \|C, V - D' - (V - H - D')\| \geq 2(4k - 3) - 8(k - 3) = 12.
\]

(c) By (b), \(\|C, H\| = 12\), so each vertex in \(H\) is heavy. Thus type(\(v\)) is the unique proper 2-coloring of \(H'\), and (c) follows. \(\square\)

Lemma 2.22. There exists \(C' \in C\) such that \(3 \leq \|\{a_1, a_2\}, C'\| \leq 4\) and \(\|\{a_1, a_2\}, D\| = 4\) for all \(D \in C - C'\). If \(\|\{a_1, a_2\}, C'\| = 3\) then one of \(a_1, a_2\) is low.

Proof. Suppose \(\|\{a_1, a_2\}, D\| \geq 5\) for some \(D \in C\); set \(H := H(D)\). Using Claim 2.21, choose notation so that \(\|\{w_1, w_3\}, H\| \leq 4\). Now

\[
\|\{w_1, w_3\}, V - H\| \geq 4k - 3 + 4 = 4(k - 2) + 1.
\]

Thus there is a cycle \(B \in C - D\) with \(\{w_1, w_3\}, B\| \geq 5\); say \(\|\{w_1, B\}\| = 3\). By Claim 2.20, \(\|C, B\| \leq 6\). Note by Claim 2.21, if \(|B| = 4\) then for an edge \(z_1z_2 \in N(w_1)\), \(w_1z_1z_2w_1\) and \(w_2w_3a_2d'w_2\) beat \(B, C\) by (O2). So \(|B| = 3\). Using Claim 2.21(b) and Lemma 2.19,

\[
2(4k - 3) \leq \|C, V\| = \|C, H \cup B \cup (V - H - B)\| \leq 12 + 6 + 8(k - 3) = 2(4k - 3).
\]

So \(\|C, D'\| = 8\) for all \(D' \in C - C - D\). By Lemma 2.19, \(\|\{w_1, w_3\}, D'\| = \|\{w_2, w_4\}, D'\| = 4\). By Claim 2.21(c) and Claim 2.20,

\[
4k - 3 \leq \|\{w_2, w_4\}, H \cup B \cup (V - H - B)\| \leq 8 + 4 + 4(k - 3) = 4k - 3,
\]

and so \(\|\{w_2, w_4\}, B\| = 1\). Say \(\|w_2, B\| = 1\). Since \(|B| = 3\), by Claim 2.12, \(G[B \cup C - w_4] \) has two disjoint cycles that are preferable to \(C, B\) by (O2). This contradiction implies \(\|\{a_1, a_2\}, D\| \leq 4\) for all \(D \in C\). Since \(\|\{a_1, a_2\}, V\| \geq 4k - 3\) and \(\|\{a_1, a_2\}, R\| = 2\), \(\|\{a_1, a_2\}, D\| \geq 3\), and equality holds for at most one \(D \in C\), and only if one of \(a_1\) and \(a_2\) is low. \(\square\)
2.4. Completion of the proof of Theorem 1.7. For an optimal $C$, let $C_i := \{D \in C : |D| = i\}$ and $t_i := |C_i|$. For $C \in C_4$, let $Q_C := Q_C(C) := G[R(C) \cup C]$. A 3-path $R'$ is $D$-useful if $R' = R(C')$ for an optimal set $C'$ with $D \subseteq C'$; we write $D$-useful for $\{D\}$-useful.

Lemma 2.23. Let $C$ be an optimal set and $C \in C_4$. Then $Q = Q_C \in \{K_3,4, K_3,4 - e\}$.

Proof. Since $C$ is optimal, $Q$ does not contain a 3-cycle. So for all $v \in V(C)$, $N(v) \cap R$ is independent and $\|a_1, C\|, \|a_2, C\| \leq 2$. By Lemma 2.22, $\|\{a_1, a_2\}, C\| \geq 3$. Say $a_1w_1, a_1w_3 \in E$ and $\|a_2, C\| \geq 1$. So type($a_1$) and type($a_2$) are defined.

Claim 2.24. type($a_1$) = type($a_2$).

Proof. Suppose not. Then $\|w_i, R\| \leq 1$ for all $i \in [4]$. Say $a_2w_2 \in E$. If $w_ia_j \in E$ and $\|a_{3-j}, C\| = 2$, let $R_i = w_ia_ja'$ and $C_i = a_{3-j}(C - w_i)a_{3-j}$ (see Figure 2.5). Then $R_i$ is $(C - C + C_i)$-useful. Let $\lambda(X)$ be the number of low vertices in $X \subseteq V$. As $Q$ does not contain a 3-cycle, $\lambda(R) + \lambda(C) \leq 2$. We claim:

$$\forall D \in C - C, \|a', D\| \leq 2.$$  

(2.8)

Fix $D \in C - C$, and suppose $\|a', D\| \geq 3$. By Claim 2.1, $|D| = 3$. Since

\[
\|C, D\| = \|C, C\| - \|C, C - D\|
\geq 4(2k - 1) - \lambda(C) - \|C, R\| - 8(k - 2)
\]

(2.9)

\[
\|w_i, D\| \geq 2 \text{ for some } i \in [4].\]

If $R_i$ is defined, $R_i$ is $\{C_i, D\}$-useful. By Lemma 2.22, $\|w_i, a', D\| \leq 4$. As $\|w_i, D\| \geq 2$, $\|a', D\| \leq 2$, proving (2.8). Then $R_i$ is not defined, so $a_2$ is low with $N(a_2) \cap C = \{w_2\}$ and $\|w_2, D\| \leq 1$. Then by (2.9), $\|C - w_2, D\| \geq 6$. Note $G[a' + D] = K_4$, so for any $z \in D$, $D - z + a'$ is a triangle, so by (O2) the neighbors of $z$ in $C$ are independent. Then $\|C - w_2, D\| = 6$ with $N(z) \cap C = \{w_1, w_3\}$ for every $z \in D$. Then $\|w_2, D\| = 1$, say $zw_2 \in E(G)$, and now $w_2w_3zw_2, w_1(D - z)w_1$ beat $C, D$ by (O2).

![Figure 2.5. Claim 2.24](image)

If $\|a', C\| \geq 1$ then $a'w_1 \in E$ and $N(a_2) \cap C = \{w_2\}$. So $R_2$ is $C_2$-useful, type($a'$) \neq type($w_2$) with respect to $C_2$, and the middle vertex $a_2$ of $R_2$ has no neighbors in $C_2$. So we may assume $\|a', C\| = 0$. Then $a'$ is low:

$$d(a') = \|a', C \cup R\| + \|a', C - C\| \leq 0 + 2 + 2(k - 2) = 2k - 2.$$ 

(2.10)

Thus all vertices of $C$ are high. Using Lemma 2.19, this yields:

$$4 \geq \|C, R\| = \|C, V - (V - R)\| \geq 4(2k - 1) - 8(k - 1) = 4.$$ 

(2.11)
As this calculation is tight, \( d(w) = 2k - 1 \) for every \( w \in C \). Thus \( d(a') \geq 2k - 2 \). So (2.10) is tight. Hence \( \|a', D\| = 2 \) for all \( D \in C - C \).

Pick \( D = z_1 \ldots z_4 z_1 \in C - C \) with \( \|(a_1, a_2), D\| \) maximum. By Lemma 2.22, \( 3 \leq \|(a_1, a_2), D\| \leq 4 \). Say \( \|(a_1, D)\| \geq 2 \). By (2.11), \( \|C, D\| = 8 \). By Lemma 2.19,

\[
W := G[C \cup D] \in \{K_{4,4}, K_3 \vee (K_3 + K_1), K_1 \vee K_{3,3}\}.
\]

**CASE 1:** \( W = K_{4,4} \). Then \( |D, R| \geq 5 > |D| = 4 \), so \( \|z, R\| \geq 2 \) for some \( z \in V(D) \). Let \( w \in N(z) \cap C \). Either \( w \) and \( z \) have a common neighbor in \( \{a_1, a_2\} \) or \( z \) has two consecutive neighbors in \( R \). Regardless, \( G[R + w + z] \) contains a 3-cycle \( D' \) and \( G[W - w - z] \) contains a 4-cycle \( C' \). Thus \( C', D' \) beats \( C, D \) by (O2).

**CASE 2:** \( W = K_3 \vee (K_3 + K_1) \). As \( \|(a', a_1), D\| \geq 4 > |D| \), there is \( z \in V(D) \) with \( D' := za'a_z \subseteq G \). Also \( W - z \) contains a 3-cycle \( C' \). So \( C', D' \) beats \( C, D \) by (O2).

**CASE 3:** \( W = K_1 \vee K_{3,3} \). Some \( v \in V(D) \) satisfies \( \|v, W\| = 6 \). There is no \( w \in W - v \) such that \( w \) has two adjacent neighbors in \( R \): else \( a \) and \( v \) would be contained in disjoint 3-cycles, contradicting the choice of \( C, D \). So \( \|w, R\| \leq 1 \) for all \( w \in W - v \), because type(a1) \( \neq \) (type(a2)). Similarly, no \( z \in D - v \) has two adjacent neighbors in \( R \). Thus

\[
2 + 3 \leq \|(a', D)\| + \|(a_1, a_2), D\| := |R, D| = \|R, D - v\| + \|R, v\| \leq 2 + 3.
\]

So \( \|(a_1, a_2), D\| = 3 \), \( R \subseteq N(v) \), and \( N(a_i) \cap K_{3,3} \) is independent. By Lemma 2.22 and the maximality of \( \|(a_1, a_2), D\| = 3; k = 3 \). Thus \( G = Y_2 \), a contradiction. \( \square \)

Returning to the proof of Lemma 2.23, we have type(a1) = type(a2). Using Lemma 2.22, choose notation so that \( a_1 w_1, a_1 w_3, a_2 w_1 \in E \). Then \( Q \) has bipartition \( \{X, Y\} \) with \( X := \{a', w_1, w_3\} \) and \( Y := \{a_1, a_2, w_2, w_4\} \). The only possible nonedges between \( X \) and \( Y \) are \( a' w_2, a' w_4 \) and \( a_2 w_3 \). Let \( C' := w_1 R w_1 \). Then \( R' := w_2 w_3 w_4 \) is \( C' \)-useful. By Lemma 2.22, \( \|(w_2, w_4), C'\| \geq 3 \). Already \( w_2, w_4 \in N(w_1) \); so because \( Q \) has no \( C_3 \), (say) \( a' w_2 \in E \). Now, let \( C'' := a_2 a' w_2 a_2 a_1 \). Then \( R'' := a_2 w_1 w_4 \) is \( C'' \)-useful; so \( \|(a_2, w_4), C''\| \geq 3 \). Again, \( Q \) contains no \( C_3 \), so \( a' w_4 \) or \( a_2 w_3 \) is an edge of \( G \). Thus \( Q \in \{K_{3,4}, K_{3,3} - e\} \). \( \square \)

**Proof of Theorem 1.7.** Using Lemma 2.23, one of two cases holds:

(C1) For some optimal set \( C \) and \( C' \in C_4 \), \( Q_{C'} = K_{3,4} - x_0 y_0 \);

(C2) for all optimal sets \( C \) and \( C' \in C_4 \), \( C[R \cup C] = K_{3,4} \).

Fix an optimal set \( C \) and \( C' \in C_4 \), where \( R = y_0 x' y \) with \( d(y_0) \leq d(y) \), such that in (C1), \( Q_{C'} = K_{3,4} - x_0 y_0 \). By Lemmas 2.22 and 2.23, for all \( C \in C_4 \), \( 1 \leq \|y_0, C\| \leq \|y, C\| \leq 2 \) and \( \|y_0, C\| = 1 \) only in Case (C1) when \( C = C' \). Put \( H := R \cup C \), \( S = S(C) := N(y) \cap H \), and \( T = T(C) := V(H) - S \). As \( \|y, R\| = 1 \) and \( \|y, C\| = 2 \) for each \( C \in C_4 \), \( |S| = 1 + 2 t_4 = |T| - 1 \).

**Claim 2.25.** \( H \) is an \( S, T \)-bigraph. In case (C1), \( H = K_{2 t_4 + 1, 2 t_4 + 2} - x_0 y_0 \); else \( H = K_{2 t_4 + 1, 2 t_4 + 2} \).

**Proof.** By Lemma 2.23, \( \|x', S\| = \|y, T\| = \|y_0, T\| = 0 \).

By Lemmas 2.22 and 2.23, \( \|y_0, S\| = |S| - 1 \) in (C1) and \( \|y_0, S\| = |S| \) otherwise. We claim that for every \( t \in T - y_0 \), \( \|t, S\| = |S| \). This clearly holds for \( y \), so take \( t \in H - \{y, y_0\} \). Then \( t \in C \) for some \( C \in C_4 \). Let \( R^* := tx_0 y_0 \) and \( C^* := y(C - t) y \). (Note \( R^* \) is a path and \( C^* \) is a cycle by Lemma 2.23 and the choice of \( y_0 \).) Since \( R^* \) is \( C^* \)-useful, by Lemmas 2.22 and 2.23, and by choice of \( y_0 \), \( \|t, S\| = \|y, S\| = |S| \). Then in (C1), \( H \supseteq K_{2 t_4 + 1, 2 t_4 + 2} - x_0 y_0 \) and \( x_0 y_0 \notin E(H) \); else \( H \supseteq K_{2 t_4 + 1, 2 t_4 + 2} \).

Now we easily see that if any edge exists inside \( S \) or \( T \), then \( C_3 + (t_4 - 1) C_4 \subseteq H \), and these cycles beat \( C_4 \) by (O2). \( \square \)
By Claim 2.25 all pairs of vertices of $T$ are the ends of a $C_3$-useful path. Now we use Lemma 2.22 to show that they have essentially the same degree to each cycle in $C_3$.

**Claim 2.26.** If $v \in T$ and $D \in C_3$ then $1 \leq \|v, D\| \leq 2$; if $\|v, D\| = 1$ then $v$ is low and for all $C \in C_3 - D$, $\|v, C\| = 2$.

**Proof.** By Claim 2.25, $H + x_0y_0$ is a complete bipartite graph. Let $y_1, y_2 \in T - v$ and $u \in S - x_0$. Then $R' = y_1uv$, $R'' = y_2uv$, and $R''' = y_1uy_2$ are $C_3$-useful. By Lemma 2.22,

$$3 \leq \|\{v, y_1\}, D\|, \|\{v, y_2\}, D\|, \|\{y_1, y_2\}, D\| \leq 4.$$ 

Say $\|y_1, D\| \leq 2 \leq \|y_2, D\|$. Thus

$$1 \leq \|\{v, y_1\}, D\| - \|y_1, D\| = \|v, D\| = \|\{v, y_2\}, D\| - \|y_2, D\| \leq 2.$$ 

Suppose $\|v, D\| = 1$. By Claim 2.25 and Lemma 2.22, for any $v' \in T - v$,

$$4k - 3 \leq \|\{v, v'\}, H \cup (C_3 - D) \cup D\| \leq 2(2t_k + 1) + 4(t_3 - 1) + 3 = 4k - 3.$$ 

Thus for all $C \in C_3 - D_0$, $\|\{v, v'\}, C\| = 4$, and so $\|v, C\| = 2$. Hence $v$ is low. $\square$

Next we show that all vertices in $T$ have essentially the same neighborhood in each $C \in C_3$.

**Claim 2.27.** Let $z \in D \in C_3$ and $v, w \in T$ with $w$ high.

1. If $zv \in E$ and $zw \notin E$ then $T - w \subseteq N(z)$.
2. $N(v) \cap D \subseteq N(w) \cap D$.

**Proof.** (1) Since $w$ is high, Claim 2.26 implies $\|w, D\| = 2$. Since $zw \notin E$, $D' := w(D - z)w$ is a 3-cycle. Let $u \in S - x_0$. Then $zvu = R(C')$ for some optimal set $C'$ with $C_3 - D + D' \subseteq C'$. By Claim 2.25, $T(C') = S + z$ and $S(C') = T - w$. If (C2) holds, then $T - w = S(C') \subseteq N(z)$, as desired. Suppose (C1) holds, so there are $x_0 \in S$ and $y_0 \in D$ with $x_0y_0 \notin E$. By Claims 2.25 and 2.26, $d(y_0) \leq (|S| - 1) + 2(t_3) = 2k - 2$, so $y_0$ is low. Since $w$ is high, $y_0 \in T - w$. But now apply Claims 2.25 and 2.26 to $T(C')$: $d(x_0) \leq |S(C')| - 1 + 2t_3 = 2k - 2$, and $x_0$ is low. As $x_0y_0 \notin E$, this is a contradiction. So $T - w = S(C') \subseteq N(z)$.

2. Suppose there exists $z \in N(v) \cap D \cap N(w)$. By (1), $T - w \subseteq N(z)$. Let $w' \in T - w$ be high. By Claim 2.26, $\|w', D\| = 2$. So there exists $z' \in N(w) \cap D \cap N(w')$ and $z \neq z'$. By (1), $T - w' \subseteq N(z')$. As $|T| \geq 4$ and at least three of its vertices are high, there exists a high $w'' \in T - w - w'$. Since $w''z, w''z' \in E$, there exists $z'' \in N(w) \cap D - N(w'')$ with $\{z, z', z''\} = V(D)$. By (1), $T - w'' \subseteq N(z'')$. Since $|T| \geq 4$ there exists $x \in T - \{w, w', w''\}$. So $\|x, D\| = 3$, contradicting Claim 2.26. $\square$

Let $y_1, y_2 \in T - y_0$ and let $x \in S$ with $x = x_0$ if $x_0y_0 \notin E$. By Claim 2.25, $y_1xy_2$ is a path, and $G - \{y_1, y_2, x\}$ contains an optimal set $C'$. Recall $y_0$ was chosen in $T$ with minimum degree, so $y_1$ and $y_2$ are high and by Claim 2.26 $\|y_i, D\| = 2$ for each $i \in [2]$ and each $D \in C_3$. Let $N = N(y_1) \cap \bigcup C_3$ and $M = \bigcup C_3 \nsubseteq N$ (see Figure 2.6). By Claim 2.25, $T$ is independent. By Claim 2.27, for every $y \in T$, $N(y) \cap \bigcup C_3 \subseteq N$, so $E(M, T) = \emptyset$. Since $y_2 \neq y_0$, also $N(y_2) \cap \bigcup C_3 = N$. 

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Claim 2.28. $M$ is independent.

Proof. First, we show (*) $\|z, S\| > t_4$ for all $z \in M$. If not then there exists $z \in D \in C_3$ with $\|z, S\| \leq t_4$. Since $\|M, T\| = \|T, T\| = 0,$

$$\|\{y_1, z\}, C_3\| \geq 4k - 3 - \|\{z, y_1\}, S\| \geq 4(t_4 + t_3 + 1) - 3 - (2t_4 + 1 + t_4) = t_4 + 4t_3 > 4t_3.$$

So there is $D' = z'z_1z_2' \in C_3$ with $\|\{z, y_1\}, D'\| \geq 5$ and $z' \in M$. As $\|y_1, D\| = 2$, $\|z, D'\| = 3.$ Since $D^* := zz_1'z_2$ is a cycle, $xy_2z_1'$ is $D^*$-useful. As $\|z_1', D^*\| = 3,$ this contradicts Claim 2.26, proving (*).

Suppose $zz' \in E(M)$; say $z \in D \in C_3$ and $z' \in D' \in C_3$. By (*) there is $u \in N(z) \cap N(z') \cap S.$ So $zz'u, y_1(D - z)y_1$ and $y_2(D' - z')y_2$ are disjoint cycles, contrary to (O1).

By Claims 2.25 and 2.28, $M$ and $T$ are independent; as remarked above $E(M, T) = \emptyset.$ So $M \cup T$ is independent. This contradicts (H3), since

$$|G| - 2k + 1 = 3t_3 + 4t_4 + 3 - 2(t_3 + t_4 + 1) + 1 = t_3 + 2t_4 + 2 = |M \cup T| \leq \alpha(G).$$

The proof of Theorem 1.7 is now complete. \qed

3. The case $k = 2$

Lovász [17] observed that any (simple or multi-) graph can be transformed into a multigraph with minimum degree at least 3, without affecting the maximum number of disjoint cycles in the graph, by using a sequence of operations of the following three types: (i) deleting a bud; (ii) suppressing a vertex $v$ of degree 2 that has two neighbors $x$ and $y$, i.e., deleting $v$ and adding a new (possibly parallel) edge between $x$ and $y$; and (iii) increasing the multiplicity of a loop or edge with multiplicity 2. Here loops and two parallel edges are considered cycles, so forests have neither. Also $K_s$ and $K_{s,t}$ denote simple graphs. Let $W_4^*$ denote a wheel on $s$ vertices whose spokes, but not outer cycle edges, may be multiple. The following theorem characterizes those multigraphs that do not have two disjoint cycles.

Theorem 3.1 (Lovász [17]). Let $G$ be a multigraph with $\delta(G) \geq 3$ and no two disjoint cycles. Then $G$ is one of the following: (1) $K_5$, (2) $W_4^*$, (3) $K_{3,|G|-3}$ together with a multigraph on the vertices of the (first) 3-class, and (4) a forest $F$ and a vertex $x$ with possibly some loops at $x$ and some edges linking $x$ to $F$.

Let $\mathcal{G}$ be the class of simple graphs $G$ with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that do not have two disjoint cycles. Fix $G \in \mathcal{G}.$ A vertex in $G$ is low if its degree is at most 2. The low vertices form a clique $Q$ of size at most 2—if $|Q| = 3$, then $Q$ is a component-cycle, and $G - Q$ has another cycle. By Lovász’s observation, $G$ can be reduced to a graph $H$ of type (1–4).
Reversing this reduction, $G$ can be obtained from $H$ by adding buds and subdividing edges. Let $Q' := V(G) \setminus V(H)$. It follows that $Q \subseteq Q'$. If $Q' \neq Q$, then $Q$ consists of a single leaf in $G$ with a neighbor of degree 3, so $G$ is obtained from $H$ by subdividing an edge and adding a leaf to the degree-2 vertex. If $Q' = Q$, then $Q$ is a component of $G$, or $G = H + Q + e$ for some edge $e \in E(H, Q)$, or at least one vertex of $Q$ subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, $e$ is subdivided twice by $Q$. As $G$ is simple, $H$ has at most one multiple edge, and its multiplicity is at most 2.

In case (4), because $\delta(H) \geq 3$, either $F$ has at least two buds, each linked to $x$ by multiple edges, or $F$ has one bud linked to $x$ by an edge of multiplicity at least 3. So this case cannot arise from $G$. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. So $Q$ is not an isolated vertex, lest deleting $Q$ leave $H$ with $\delta(H) \geq 5 > 4$; and if $Q$ has a vertex of degree 1 then $H = K_5$. Else all vertices of $Q$ have degree 2, and $Q$ consists of the subdivision vertices of one edge of $H$. So we have the following lemma.

**Lemma 3.2.** Let $G$ be a graph with $|G| \geq 6$ and $\sigma_2(G) \geq 5$ that does not have two disjoint cycles. Then $G$ is one of the following (see Figure 3.1):

(a) $K_5 + K_2$;
(b) $K_5$ with a pendant edge, possibly subdivided;
(c) $K_5$ with one edge subdivided and then a leaf added adjacent to the degree-2 vertex;
(d) a graph $H$ of type (1–3) with no multiple edge, and possibly one edge subdivided once or twice, and if $|H| = 6 - i$ with $i \geq 1$ then some edge is subdivided at least $i$ times;
(e) a graph $H$ of type (2) or (3) with one edge of multiplicity two, and one of its parallel parts is subdivided once or twice—twice if $|H| = 4$.

**Figure 3.1.** Theorem 3.2

4. Connections to Equitable Coloring

A proper vertex coloring of a graph $G$ is **equitable** if any two color classes differ in size by at most one. In 1970 Hajnal and Szemeredi proved:

**Theorem 4.1** ([7]). *Every graph $G$ with $\Delta(G) + 1 \leq k$ has an equitable $k$-coloring.*

For a shorter proof of Theorem 4.1, see [14]; for an $O(k|G|^2)$-time algorithm see [13]. Motivated by Brooks’ Theorem, it is natural to ask which graphs $G$ with $\Delta(G) = k$ have equitable $k$-colorings. Certainly such graphs are $k$-colorable. Also, if $k$ is odd then $K_{k,k}$ has
no equitable $k$-coloring. Chen, Lih, and Wu [2] conjectured (in a different form) that these are the only obstructions to an equitable version of Brooks’ Theorem:

**Conjecture 4.2** ([2]). If $G$ is a graph with $\chi(G), \Delta(G) \leq k$ and no equitable $k$-coloring then $k$ is odd and $K_{k,k} \subseteq G$.

In [2], Chen, Lih, and Wu proved Conjecture 4.2 holds for $k = 3$. By a simple trick, it suffices to prove the conjecture for graphs $G$ with $|G| = ks$. Combining the results of the two papers [11] and [12], we have:

**Theorem 4.3.** Suppose $G$ is a graph with $|G| = ks$. If $\chi(G), \Delta(G) \leq k$ and $G$ has no equitable $k$-coloring, then $k$ is odd and $K_{k,k} \subseteq G$ or both $k \geq 5$ [11] and $s \geq 5$ [12].

A graph $G$ is $k$-equitable if $|G| = ks$, $\chi(G) \leq k$ and every proper $k$-coloring of $G$ has $s$ vertices in each color class. The following strengthening of Conjecture 4.2, if true, provides a characterization of graphs $G$ with $\chi(G), \Delta(G) \leq k$ that have an equitable $k$-coloring.

**Conjecture 4.4** ([10]). Every graph $G$ with $\chi(G), \Delta(G) \leq k$ has an no equitable $k$-coloring if and only if $k$ is odd and $G = H + K_{k,k}$ for some $k$-equitable graph $H$.

The next theorem collects results from [10]. Together with Theorem 4.3 it yields Corollary 4.6.

**Theorem 4.5** ([10]). Conjecture 4.2 is equivalent to Conjecture 4.4. Indeed, for any $k_0$ and $n_0$, Conjecture 4.2 holds for $k \leq k_0$ and $|G| \leq n_0$ if and only if Conjecture 4.4 holds for $k \leq k_0$ and $|G| \leq n_0$.

**Corollary 4.6.** A graph $G$ with $|G| = 3k$ and $\chi(G), \Delta(G) \leq k$ has no equitable $k$-coloring if and only if $k$ is odd and $G = K_{k,k} + K_k$.

We are now ready to complete our answer to Dirac’s question for simple graphs.

**Proof of Theorem 1.3.** Assume $k \geq 2$ and $\delta(G) \geq 2k - 1$. It is apparent that if any of (i), (H3), or (H4) in Theorem 1.3 fail, then $G$ does not have $k$ disjoint cycles. Now suppose $G$ satisfies (i), (H3), and (H4). If $k = 2$ then $|G| \geq 6$ and $\delta(G) \geq 3$. Thus $G$ has no subdivided edge, and only (d) of Lemma 3.2 is possible. By (i), $G \not\subseteq K_5$; by (H4), $G$ is not a wheel; and by (H3), $G$ is not type (3) of Theorem 3.1. So $G$ has 2 disjoint cycles. Finally, suppose $k \geq 3$. Since $G$ satisfies (ii), $G \not\subseteq \{Y_1, Y_2\}$ and $G$ satisfies (H2). So, if $|G| \geq 3k + 1$ then $G$ has $k$ disjoint cycles by Theorem 1.7. Otherwise, $|G| = 3k$ and $G$ has $k$ disjoint cycles if and only if its vertices can be partitioned into disjoint $K_3$’s. This is equivalent to $\overline{G}$ having an equitable $k$-coloring. By (ii), $\Delta(\overline{G}) \leq k$, and by (H3), $\omega(\overline{G}) \leq k$. So by Brooks’ Theorem, $\chi(\overline{G}) \leq k$. By (H4) and Corollary 4.6, $\overline{G}$ has an equitable $k$-coloring. $\square$

Next we turn to Ore-type results on equitable coloring. To complement Theorem 1.7, we need a theorem that characterizes when a graph $G$ with $|G| = 3k$ that satisfies (H2) and (H3) has $k$ disjoint cycles, or equivalently, when its complement $\overline{G}$ has an equitable coloring. The complementary version of $\sigma_2(G)$ is the maximum Ore-degree $\theta(H) := \max_{xy \in E(H)}(d(x) + d(y))$. So $\theta(\overline{G}) = 2|G| - \sigma_2(G) - 2$, and if $|G| = 3k$ and $\sigma_2(G) \geq 4k - 3$ then $\theta(\overline{G}) \leq 2k + 1$. Also, if $G$ satisfies (H3) then $\omega(\overline{G}) \leq k$. This would correspond to an Ore-Brooks-type theorem on equitable coloring.

Several papers, including [8, 9, 16], address equitable colorings of graphs $G$ with $\theta(G)$ bounded from above. For instance, the following is a natural Ore-type version of Theorem 4.1.
Theorem 4.7 ([8]). Every graph $G$ with $\theta(G) \leq 2k - 1$ has an equitable $k$-coloring.

Even for ordinary coloring, an Ore-Brooks-type theorem requires forbidding some extra subgraphs when $\theta$ is 3 or 4. It was observed in [9] that for $k = 3, 4$ there are graphs for which $\theta(G) \leq 2k + 1$ and $\omega(G) \leq k$, but $\chi(G) \geq k + 1$. The following theorem was proved for $k \geq 6$ in [9] and then for $k \geq 5$ in [16].

Theorem 4.8. Let $k \geq 5$. If $\omega(G) \leq k$ and $\theta(G) \leq 2k + 1$, then $\chi(G) \leq k$.

In a forthcoming paper with T. Molla we prove an analog of Theorem 1.7 for $3k$-vertex graphs.

References