Edge coloring multigraphs without small dense subsets

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Abstract

One consequence of an old conjecture of Goldberg and Seymour about the chromatic index of multigraphs would be the following statement. Suppose \(G\) is a multigraph with maximum degree \(\Delta\), such that no vertex subset \(S\) of odd size at most \(\Delta\) induces more than \((\Delta + 1)(|S| - 1)/2\) edges. Then \(G\) has an edge coloring with \(\Delta + 1\) colors. Here we prove a weakened version of this statement.

1 Introduction

In this note we study edge colorings of (loopless) multigraphs. We use the standard notation \(\chi'(G)\) to denote the chromatic index of \(G\), that is, the smallest number of matchings needed to partition the edge set of \(G\). The maximum degree of \(G\) is denoted by \(\Delta(G)\). The classical upper bounds for \(\chi'(G)\) are \(\chi'(G) \leq 3\Delta(G)/2\) (Shannon’s Theorem [13]) and \(\chi'(G) \leq \Delta + \mu(G)\) (Vizing’s Theorem [16]), where \(\mu(G)\) denotes the maximum edge multiplicity of \(G\).

For a subset \(S \subseteq V(G)\) of the vertices of a multigraph \(G\), we denote by \(G[S]\) the subgraph induced by \(S\), by \(|S|\) the number of edges in \(G[S]\), and by \(\rho(S)\) the quantity \(\frac{|S|}{\lceil |S|/2 \rceil}\). The parameter \(\rho(G)\) is defined by

\[
\rho(G) = \max\{\rho(S) : S \subseteq V(G)\}.
\]

Then \(\lceil \rho(G) \rceil\) is a lower bound on \(\chi'(G)\), since for a set \(S\) on which \(\rho(G)\) is attained, each matching in \(G[S]\) has size at most \(\lceil |S|/2 \rceil\) and therefore at least \(\lceil \frac{|S|}{\lceil |S|/2 \rceil} \rceil\) colors

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are needed to color the edges of $G[S]$. Another natural lower bound on $\chi'(G)$ is given by the maximum degree $\Delta(G)$. A long-standing conjecture due to Goldberg [3] (see also [4]) and independently Seymour [12] states that the chromatic index of $G$ should be essentially determined by either $\rho(G)$ or $\Delta(G)$.

**Conjecture 1** For every multigraph $G$

$$\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \rho(G) \rceil\}.$$ 

Goldberg [4] also proposed the following sharp version for multigraphs with $\rho(G) \leq \Delta(G) - 1$.

**Conjecture 2** For every multigraph $G$, if $\rho(G) \leq \Delta(G) - 1$ then $\chi'(G) = \Delta(G)$.

Conjecture 1 implies that if $\chi'(G) > \Delta + k$, $k \geq 1$, then $G$ must contain a set $S$ of vertices for which $\rho(S) > \Delta + k$, certifying this inequality. Thus $S$ induces a very dense subgraph in $G$. It is easy to verify that $|S|$ is odd and is at most $\Delta$ for such $S$, in particular $S$ is small in the sense that its size depends only on $\Delta$ and not on the number of vertices of $G$. Conjecture 2 gives a similar statement for $k = 0$, but the corresponding set $S$ need not be small.

We can therefore think of Conjecture 1 as providing structural information about multigraphs for which $\chi'(G) > \Delta + 1$, namely, that they must contain small sets $S$ that are very dense. Our aim in this note is to prove a result of this form. Unfortunately we cannot make such a conclusion about all $G$ with $\chi'(G) > \Delta + 1$, but we show that when $k$ is bounded below by a logarithmic function of $\Delta$ then a structural result of this type for multigraphs $G$ satisfying $\chi'(G) > \Delta + k$ is possible.

Conjecture 1 has inspired a significant body of work, with contributions from many researchers, see for example [14] or [6] for an overview. Here we mention just the results that directly relate to this note. The best known approximate version is as follows, due to Scheide [9] (independently proved by Chen, Yu and Zang [1], see also [10] and [2]).

**Theorem 3** For every multigraph $G$

$$\chi'(G) \leq \max\{\Delta(G) + \sqrt{\frac{\Delta(G) - 1}{2}}, \lceil \rho(G) \rceil\}.$$ 

Since $\lceil \rho(S) \rceil > \Delta + \sqrt{\frac{\Delta - 1}{2}}$ implies $|S| < \sqrt{\frac{2\Delta^2}{\Delta - 1}} + 1$, the following theorem about multigraphs without small dense subsets is implied by Theorem 3.
Corollary 4 Let $G$ be a multigraph with maximum degree $\Delta$. If $\lceil \rho(S) \rceil \leq \Delta + \sqrt{\frac{\Delta - 1}{2}} - 1$ for every $S \subseteq V(G)$ with $|S| < \sqrt{\frac{2\Delta}{\Delta - 1}} + 1$ then $\chi'(G) \leq \Delta + \sqrt{\frac{\Delta - 1}{2}}$.

The main theorem of this note states that if the density of small vertex subsets $S$ is restricted somewhat further then a substantially better upper bound can be given for $\chi'(G)$, in which the quantity $\sqrt{\frac{\Delta - 1}{2}}$ from Corollary 4 is replaced by a logarithmic function of $\Delta$. It can also be viewed as a weakened version of the statement of Conjecture 2.

Theorem 5 Let $G$ be a multigraph with maximum degree $\Delta$, and let $\varepsilon$ be given where $0 < \varepsilon < 1$. Let $k = \lfloor \log_{1+\varepsilon}(\Delta) \rfloor$. If $\rho(S) \leq (1 - \varepsilon)(\Delta + k)$ for every $S \subseteq V(G)$ with $|S| < \Delta/k + 1$ then $\chi'(G) \leq \Delta + k$.

For example, this implies that $\chi'(G) < \Delta + 101 \log \Delta$ unless $G$ contains a set $S$ of vertices with $|S| < \frac{\Delta}{100 \log \Delta}$ with density parameter $\rho(S) > 0.99(\Delta + 100 \log \Delta)$.

Our proof uses the technique of Tashkinov trees, developed by Tashkinov in [15]. In the next section we give a brief introduction to this technique together with the main tools we use, including our main technical lemma, Lemma 8. The proof of Theorem 5 appears in Section 3.

2 Tools

The method of Tashkinov trees, due to Tashkinov [15], is a sophisticated generalization of the method of alternating paths. It is based on an earlier approach from [7]. See [14] for a comprehensive account of this technique.

Let $G$ be a multigraph with $\chi'(G) \geq \Delta + 2$, and let $\phi$ be a partial edge coloring of $G$ that uses at most $\chi' - 1$ colors. We say $\phi$ is a $t$-coloring if the set of colors used by $\phi$ is $\{1, \ldots, t\}$. We normally assume $\phi$ is maximal, that is, the maximum possible number of edges of $G$ are colored by $\phi$. For a vertex $v$ of $G$, color $\alpha$ is said to be missing at $v$ if no edge incident to $v$ is colored $\alpha$ by $\phi$. Let $T = (p_0, e_0, p_1, \ldots, p_n)$ be a sequence of distinct vertices $p_i$ and edges $e_i$ of $G$, such that the vertices of each $e_i$ are $p_{i+1}$ and $p_r$ for some $r \in \{0, \ldots, i\}$. Observe then that $T$ is a tree. We say that $T$ is a Tashkinov tree with respect to $\phi$ if $e_0$ is uncolored, and for all $i > 0$, the color $\phi(e_i)$ is missing at $p_j$ for some $j < i$. Thus $T$ is a Tashkinov tree if its first edge is uncolored, and each subsequent edge is colored with a color that is missing at some previous vertex. The key property of Tashkinov trees is captured in the following theorem, due to Tashkinov [15].

Theorem 6 Let $\phi$ be a maximal partial edge coloring of $G$ with at most $\chi'(G) - 1$ colors, and let $T$ be a Tashkinov tree with respect to $\phi$. Then no two vertices of $T$ are missing the same color.
For a color \( \omega \) we denote by \( \partial_\omega(T) \) the set of edges of color \( \omega \) that have exactly one vertex in \( T \). If \( \omega \) is missing on a vertex of \( T \) we set \( q_\omega(T) = |\partial_\omega(T)| + 1 \), and \( q_\omega(T) = |\partial_\omega(T)| \) otherwise. (Thus \( q_\omega(T) \) counts the number of vertices in \( T \) that are not incident with an edge of \( G[T] \) of color \( \omega \).) Then the following is an immediate corollary of Theorem 6.

**Corollary 7** Let \( \phi \) be a maximal partial edge coloring of \( G \) with at most \( \chi'(G) - 1 \) colors, and let \( T \) be a Tashkinov tree with respect to \( \phi \). If \( |T| \) is odd then \( q_\omega(T) \) is odd for every color \( \omega \).

Let \( T \) be a Tashkinov tree with respect to some maximal coloring \( \phi \). If a color \( \alpha \) is missing on \( v \in T \) and not used by \( \phi \) on an edge of \( T \) we say that \( \alpha \) is free for \( T \). The number of colors missing at \( v \) that are free for \( T \) is denoted by \( f_T(v) \), or simply \( f(v) \) if there is no danger of confusion. We set \( f(T) = \min\{f(v) : v \in T \} \).

It was observed by e.g. [2] that if \( T \) is a Tashkinov tree with respect to \( \phi \) such that \( \rho(G) \) is not attained on \( V(T) \), and if \( f(T) > 0 \), then by (possibly) replacing \( \phi \) by another maximal coloring it is possible to construct a Tashkinov tree that is larger than \( T \). This technical fact was used in several results using Tashkinov trees, for example [1, 5, 9]. Our main lemma, Lemma 8, is also based on this parameter.

**Lemma 8** Let \( G \) be a multigraph with maximum degree \( \Delta \) and suppose \( \chi'(G) \geq \Delta + 2 \). Let \( \phi \) be a maximal \((\Delta + k)\)-coloring of \( G \), where \( \Delta + k \leq \chi'(G) - 1 \), and let \( T \) be a Tashkinov tree with respect to \( \phi \) such that \( f(T) > 0 \). Let \( \omega \in \{1, \ldots, \Delta + k\} \) be a color. Then there exists a maximal \((\Delta + k)\)-coloring \( \psi \), and a Tashkinov tree \( T' \) with respect to \( \psi \) such that

1. \( T \subset T' \)
2. \( f(T') \geq f(T) - 1 \)
3. \( |T'| \geq |T| + q_\omega(T) - 1 \).

**Proof.** If \( \omega \) is missing on a vertex of \( T \) then we may simply add the edges in \( \partial_\omega(T) \) to \( T \), forming \( T' \) with \( |T'| = |T| + q_\omega(T) - 1 \) and \( f(T') \geq f(T) - 1 \), since only one color is used on \( E(T') \setminus E(T) \). Then \( \psi = \phi \) satisfies the theorem.

We may therefore assume that \( \omega \) is not missing on \( T \). Set \( q = q_\omega(T) = |\partial_\omega(T)| \).

Let \( \gamma \) be a color missing on some vertex \( v \in T \) for which \( f(v) \geq k \). This is possible by Theorem 6 together with the observation that at most \( |T| - 2 \) colors are used on \( T \). We consider the \((\gamma, \omega)\)-alternating path \( P \) beginning at \( v \). The other end \( z \) of \( P \) is not a vertex of \( T \), since \( \omega \) is not missing in \( T \) and by Theorem 6 no \( x \in T \) different from \( v \) can be missing \( \gamma \). Let \( y \) be the last vertex of \( P \) in \( T \) and denote by \( Q \) the \((y, z)\)-segment of \( P \). Then \( E(Q) \cap E(T) = \emptyset \). Since \( f(T) > 0 \) there exists a color \( \alpha \) missing on \( y \) that is not used on \( T \).
For \( i \geq 0 \) we now define a sequence of Tashkinov trees \( T_i \) with respect to \( \phi \), together with colors \( \alpha_i \), vertices \( z_i \) and segments \( Q_i \) of \( Q \) satisfying the following properties.

1. \( T_0 \subset \cdots \subset T_i \),
2. \( \alpha_i \) is missing on \( z_i \) and not used on \( T_{i+1} \),
3. \( f(T_i) \geq f(T) - 1 \) for each \( i \geq 1 \),
4. for \( i \geq 1 \), every edge of \( E(T_i) \setminus E(T_{i-1}) \) is of color \( \gamma \) or \( \alpha_{i-1} \),
5. \( Q_i \) is the \((z_i, z)\)-segment of \( Q \), and for \( i \geq 1 \) the length of \( Q_i \) is positive but less than the length of \( Q_{i-1} \).

We begin the construction by setting \( T_0 = T \), \( \alpha_0 = \alpha \), \( z_0 = y \), and \( Q_0 = Q \). Then (1)-(5) hold for \( i = 0 \).

Suppose \( i \geq 0 \) and that we have completed the construction up to \( i \). We now consider two cases according to whether any \((\alpha_i, \gamma)\)-component intersects both \( T_i \) and \( E(Q_i) \). If there is such a component then we show that either \( \phi \) itself satisfies the theorem, or that we can extend our sequence. If no such component exists then we will terminate the sequence and find a recoloring \( \psi \) that satisfies the theorem.

**Case 1:** Some \((\alpha_i, \gamma)\)-component \( R \) contains an edge of \( Q_i \) and a vertex of \( T_i \).

In this case we define \( T_{i+1} \) to be the Tashkinov tree obtained by adding \( R \) to \( T_i \). This is a valid tree for \( \phi \) because \( \alpha_i \) and \( \gamma \) are both missing on \( T_i \). Then (1) and (4) are satisfied for \( i + 1 \). We let \( z_{i+1} \) be the vertex of \( T_{i+1} \) that is closest to \( z \) on \( Q_i \), and note that the \((z_{i+1}, z)\)-segment \( Q_{i+1} \) is shorter than \( Q_i \) because \( R \) contained an edge of \( Q_i \), verifying the second condition in (5) for \( i + 1 \). Let \( \alpha_{i+1} \) be any color missing on \( z_{i+1} \), then (2) is satisfied for \( i + 1 \).

To verify Condition (3) for \( i + 1 \), observe that by (4), every edge of \( E(T_{i+1}) \setminus E(T) \) has one of the colors \( \gamma \) or \( \alpha_j \) for some \( 0 \leq j \leq i \). By (2), the colors \( \alpha_i \) for \( i \geq 1 \) are missing on the vertices \( z_i \), and since the \( z_i \) are all distinct (by (5)), no other color missing on \( z_i \) is used on \( T_{i+1} \). By choice of \( \gamma_i \), which is missing on \( v \), we know \( f_{T_{i+1}}(v) > f(T) - 1 \) since no other colors missing on \( v \) were used. Therefore the only new color used that may affect \( f(T_{i+1}) \) is \( \alpha_0 = \alpha \), and hence \( f(T_{i+1}) \geq f(T) - 1 \).

Finally we turn to the first condition in (5). If this condition holds, in other words \( z_{i+1} \neq z \), then we extend our sequence using the above definitions. If \( z_{i+1} = z \), then we claim that \( \phi \) satisfies the theorem in this case. Note that if \( \gamma \) is missing at \( z \) then we have a contradiction to Theorem 6, because \( \gamma \) is also missing at \( v \neq z \). Therefore \( \omega \) is missing at \( z \). Then we may construct \( T' \) by adding all remaining edges of \( \partial_\omega(T) \) that join a vertex of \( T_{i+1} \) to a vertex outside \( T_{i+1} \). By the existence of \( R \) this in fact gives us \( |T'| \geq |T| + q \). By (3) for \( i + 1 \) we have \( f(T_{i+1}) \geq f(T) - 1 \), and the only new
color used in the construction of $T'$ from $T_{i+1}$ is $\omega$, which is missing on $z_{i+1}$. But no other color missing on $z_{i+1}$ appears on an edge of $T'$, so $\omega$ does not contribute to $f(T')$. Thus $f(T') \geq f(T) - 1$.

**Case 2:** No $(\alpha_i, \gamma)$-component contains an edge of $Q_i$ and a vertex of $T_i$.

In this case we modify $\phi$. First we interchange $\alpha_i$ and $\gamma$ on every $(\alpha_i, \gamma)$-component containing an edge of $Q_i$. Since we are in Case 2, this change does not affect the color of any edge induced by $V(T_i)$. Therefore $T_i$ is a Tashkinov tree with respect to the new coloring. The path $Q_i$ becomes an $(\alpha_i, \omega)$-path from $z_i$ to $z$, which (as before) is disjoint from all of $\partial_\omega(T)$ except the $\omega$-edge $e$ incident to $y$. We complete the construction of $\psi$ by interchanging $\omega$ and $\alpha_i$ on $Q_i$. Then $\omega$ is missing on $z_i$. We construct $T'$ by adding to $T_i$ all the edges of $\partial_\omega(T) \setminus \{e\}$ that join $V(T_i)$ to its complement. Then $|T'| \geq |T| + q - 1$ (and if $i \geq 1$ then $|T'| \geq |T| + q$). The only new color used that was not used on $T_i$ is $\omega$, which is missing on $z_i$. If $i = 0$ then trivially $f(T') \geq f(T) - 1$. If $i \geq 1$ then by (3) and the fact that no other color on $z_i$ is used on $T'$ we have $f(T') \geq f(T) - 1$. This completes the proof. \(\square\)

## 3 Proof of Theorem 5

The proof of Theorem 5 follows by a sequence of applications of Lemma 8.

**Proof.** The theorem is trivially true for $\Delta = 1$ so we may assume $\Delta \geq 2$, and hence $k \geq 1$. If $\chi'(G) \leq \Delta + k$ then the conclusion of the theorem holds so we may assume on the contrary that $\Delta + k \leq \chi'(G) - 1$. Let $\phi$ be a maximal $(\Delta + k)$-coloring of $G$. Since $\chi'(G) > \Delta + k$, there is an uncolored edge $e_0$ with vertices $p_0$ and $p_1$.

For the proof of Theorem 5 we provide a construction consisting of a series of steps. We begin with the partial coloring $\psi_1 = \phi$. At each step $i \geq 2$ an application of Lemma 8 is used to construct a new maximal $(\Delta + k)$-coloring $\psi_i$ with $e_0$ uncolored and a new Tashkinov tree $T_i$ with $|T_i| \geq 1 + (1 + \varepsilon)^i$, where $|T_i|$ is odd.

**Step 1.** Set $\psi_1 = \phi$ and let $p_2$ be a vertex joined to $p_1$ by an edge whose color $\alpha$ is missing at $p_0$. Then $\{p_0, p_1, p_2\}$ forms a Tashkinov tree $T_1$ with respect to $\psi_1$, and $f(T_1) \geq k$ because there were at most $\Delta - 1$ colored edges incident to $p_0$. Note that $|T_1| = 3 \geq 1 + (1 + \varepsilon)$, since $\varepsilon < 1$.

**Step i.** Suppose that the Tashkinov tree $T_{i-1}$ and coloring $\psi_{i-1}$ have been defined for some $2 \leq i \leq k + 1$, such that $f(T_{i-1}) \geq k - i + 2$, $|T_{i-1}| \geq 1 + (1 + \varepsilon)^{i-1}$, and $|T_{i-1}|$ is odd. Choose a color $\omega$ such that $q = q_\omega(T_{i-1})$ is largest. Since $|T_{i-1}|$ is odd, we know by Corollary 7 that $q$ is odd. Consider two cases:

**Case 1:** $q = q_\omega(T_{i-1}) \leq \varepsilon |T_{i-1}| + 1 - \varepsilon$.

Then each color occurs on at least $(|T_{i-1}| - q)/2 \geq (1 - \varepsilon)(|T_{i-1}| - 1)/2$ edges of $T_{i-1}$. As $e_0 \in T_{i-1}$ is uncolored,

$$|T_{i-1}| \geq (\Delta + k)(1 - \varepsilon)(|T_{i-1}| - 1)/2 + 1.$$
Therefore $S = V(T_{i-1})$ is such that

$$\rho(S) > (1 - \varepsilon)(\Delta + k).$$

Moreover $|S| < \Delta/k + 1$ by Theorem 6, because at least $k|S| + 2$ colors are missing on the vertices of $S$. This contradicts the assumptions of Theorem 5.

**Case 2:** $q = q(T_{i-1}) > \varepsilon|T_{i-1}| + 1 - \varepsilon$.

As $|T_{i-1}| \geq 3$ and $q$ is an odd integer, $q \geq 3$. Let $\psi_i$ be the maximal coloring and $T'$ be the Tashkinov tree given by Lemma 8. Then by that lemma $f(T') \geq k - i + 1$, and

$$|T'| \geq |T_{i-1}| + q - 1 > 1 + (|T_{i-1}| - 1)(1 + \varepsilon) \geq 1 + (1 + \varepsilon)^i.$$  

If $|T'|$ is odd (e.g. if $|T'| = |T_i| + q - 1$) then we set $T_i = T'$. If $|T'|$ is even then choose an arbitrary color $\beta$ that is used by $\psi_i$ on an edge of $T'$. Then Theorem 6 implies that some edge $e$ colored $\beta$ has exactly one vertex in $T'$. We define $T_i$ to be the Tashkinov tree formed by adding $e$ to $T'$, so that $|T_i|$ is odd and $f(T_i) = f(T') \geq k - i + 1$.

It suffices to show that eventually Case 1 occurs. Otherwise, we construct a maximal coloring $\psi_{k+1}$ and a Tashkinov tree $T_{k+1}$ with $|T_{k+1}| \geq 1 + (1 + \varepsilon)^{k+1}$. By Theorem 6 this implies that the number of colors that are missing on the vertices of $T_k$ is at least $k(1 + (1 + \varepsilon)^{k+1}) + 2$. Then using the definition of $k$ we derive the contradiction

$$\Delta + k > k(1 + (1 + \varepsilon)^{k+1}) > k + k\Delta.$$

$\square$

For each $0 < \varepsilon < 1$ and $k = \lfloor \log_{1+\varepsilon} \Delta \rfloor$, the proof of Theorem 5 shows the existence of either an edge coloring of $G$ with $\Delta + k$ colors or a small, dense set $S$ with $|S| \leq \Delta/k + 1$ and $\rho(S) > (1 - \varepsilon)(\Delta + k)$. In fact this yields a procedure for constructing one of these structures in time polynomial in $|E(G)|$. We start by greedily coloring the edges of $G$ with colors $\{1, \ldots, \Delta + k\}$. If we get stuck before finishing then as in the proof of Theorem 5 we attempt to construct a large Tashkinov tree $T$. If we halt in Case 1 then we have constructed a small, dense set $S = V(T)$. Otherwise at some point in Case 2, some color is missed at distinct vertices of $T$. In this case, the proof of Theorem 6 (which gives a polynomial time algorithm [15], see e.g. [8] or [11]) allows us to recolor $G$ so that there is an additional colored edge. Then we start over using this new coloring. After fewer than $|E(G)|$ restarts our procedure halts with a small, dense subset or a proper edge coloring with $\Delta + k$ colors.

**References**


[10] D. Scheide, A polynomial-time $\Delta + \sqrt{\Delta^2 - 4\Delta - 4}$-edge coloring algorithm, Preprints 2009 No. 4, IMADA, The University of Southern Denmark, 15 pages.


