ON THE CHOICE NUMBER OF COMPLETE MULTIPARTITE GRAPHS WITH PART SIZE FOUR

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ABSTRACT. Let $\mathrm{ch}(G)$ denote the choice number of a graph G, and let K_{s*k} be the complete k-partite graph with s vertices in each part. Erdős, Rubin, and Taylor showed that $\mathrm{ch}(K_{2*k}) = k$, and suggested the problem of determining the choice number of K_{s*k}. The first author established $\mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$. Here we prove $\mathrm{ch}(K_{4*k}) = \lceil \frac{3k-1}{2} \rceil$.

1. INTRODUCTION

Let $G = (V, E)$ be a graph. A *list assignment* L for G is a function $L : V \to 2^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers and $2^{\mathbb{N}}$ is the power set of $\mathbb{N}$. If $|L(v)| = k$ for all vertices $v \in V$, then L is a *k-list assignment* for G. An *L-coloring* f from a list assignment L is a function $f : V \to \mathbb{N}$ such that $f(v) \in L(v)$ for all vertices $v \in V$ and $f(x) \neq f(y)$ whenever $xy \in E$. Graph G is *L-colorable* if there exists an L-coloring of G; it is *k-choosable* if it is L-colorable for all k-list assignments L. The *list chromatic number* or *choice number* of G, denoted $\mathrm{ch}(G)$, is the smallest integer k such that G is k-choosable. The general list coloring problem may consider list assignments with uneven list sizes.

The study of list coloring was initiated by Vizing [17] and by Erdős, Rubin and Taylor [5]. It is a generalization of two well studied areas of combinatorics—graph coloring and transversal theory. Restricting the list assignment to a constant function, yields ordinary graph coloring; restricting the graph to a clique yields the problem of finding a system of distinct representatives (SDR) for the family of lists. Both restrictions play a role in this paper. Given the general nature of this parameter, it is hardly surprising that there are not many graphs whose exact choice number is known. However, there are some amazingly elegant results that add to the subject's charm. For example, Thomassen [16] proved that planar graphs have choice number at most 5, Voigt [19] proved that this is tight, and Galvin [6] proved that line graphs of bipartite graphs have choice number equal to their clique number.

Erdős et al. [5] suggested determining the choice number of uniform complete multipartite graphs. More generally, let $K_{1*k_1,2*k_2\ldots}$ denote the complete multipartite graph with k_i parts of size i, where zero terms in the subscript are deleted. Since K_{1*k} is a clique and K_{s*1} is an independent set, these cases are trivial. Alon [1] proved the general bounds $c_1 k \log s \leq \mathrm{ch}(K_{s*k}) \leq c_2 k \log s$ for some constants $c_1, c_2 > 0$. This was tightened by Gazit and Krivelevich [7].

Theorem 1 (Gazit and Krivelevich [7]). $\mathrm{ch}(K_{s*k}) = (1 + o(1))\frac{\log s}{\log(1+1/k)}$.

The next well-known example provides the best lower bounds for small values of s.

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Example 2. $\mathrm{ch}(K_{s*k}) \geq l = \lceil \frac{2(s-1)k-s+2}{s} \rceil$: Let $G = K_{s*k}$ have parts $\{X_1, \dots, X_k\}$ with $X_i = \{v_{i,1}, \dots, v_{i,s}\}$. We will construct an $(l-1)$-list assignment L from which G cannot be colored. Equitably partition $C := [2k-1]$ into s parts $C_1, \dots, C_s$. Define a list assignment L for G by $L(v_{i,j}) = C \smallsetminus C_j$. Then each list has size at least

$$2k - 1 - \left\lceil \frac{2k-1}{s} \right\rceil = \left\lfloor \frac{2ks - s - 2k + 1}{s} \right\rfloor = \left\lceil \frac{2(s-1)k - 2s + 2}{s} \right\rceil = l - 1.$$

Consider any color $\alpha \in C$. Then $\alpha \in C_i$ for some $i \in [s]$. So $\alpha \notin L(v_{i,j})$ for every $j \in [k]$. Thus any L-coloring of G uses at least two colors for every part X_j. Since vertices in distinct parts are adjacent, they require distinct colors. As there are k parts this would require $2k > |C|$ colors, which is impossible.

Restricting the question of Erdős et al., we ask for those integers s such that:

$$(1.1) \qquad (\forall k \in \mathbb{Z}^+) \left[\mathrm{ch}(K_{s*k}) = l(s,k) := \left\lceil \frac{2(s-1)k - s + 2}{s} \right\rceil \right].$$

The first two cases $s = 2$ and $s = 3$ have been solved:

Theorem 3 (Erdős, Rubin and Taylor [5]). *All positive integers k satisfy* $\mathrm{ch}(K_{2*k}) = k$.

Theorem 4 (Kierstead [8]). *All positive integers k satisfy* $\mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$.

Recently, Kozik, Micek, and Zhu [9] gave a very different proof of Theorem 4. The following more general result appears in [12].

Theorem 5 (Ohba [12]). $\mathrm{ch}(K_{1*k_1,3*k_3}) = \max\{k, \lceil \frac{n+k-1}{3} \rceil\}$, *where* $k = k_1 + k_3$ *and* $n = k_1 + 3k_3$.

The next example shows that the largest s satisfying (1.1) is at most 14.

Example 6. If k is even then $\mathrm{ch}(K_{15*k}) \geq l := 2k$: Let $G = K_{15*k}$ have parts $\{X_1, \dots, X_k\}$ with $X_i = \{v_{i,1}, \dots, v_{i,15}\}$. We will construct an $(l-1)$-list assignment L from which G cannot be colored. Equitably partition $C := [3k-1]$ into 6 parts $C_1, \dots, C_6$, and fix a bijection $f : [15] \to \binom{[6]}{2}$. Define a list assignment L for G by

$$L(v_{i,j}) = C \smallsetminus \bigcup \{C_h : h \in f(i)\}.$$

Then each list has size at least

$$3k - 1 - 2\left\lceil \frac{3k-1}{6} \right\rceil = 2k - 1 = l - 1.$$

Consider any two colors $\alpha, \beta \in C$. Then $\alpha, \beta \in \bigcup \{C_h : h \in f(i)\}$ for some $i \in [15]$. So $\alpha, \beta \notin L(v_{i,j})$ for every $j \in [k]$. Thus any L-coloring of G uses at least three colors for every part X_j. Since $3k > |C|$, this is impossible.

Yang [20] proved $\lceil \frac{3k}{2} \rceil \leq \mathrm{ch}(K_{4*k}) \leq \lceil \frac{7k}{4} \rceil$, and Noel, West, Wu and Zhu [11] improved the upper bound to $\lceil \frac{5k-1}{3} \rceil$. The main result of this paper is that (1.1) holds for $s = 4$. To prove this theorem we first extract a simple proof of Theorem 4 from [11], and then elaborate on it.

Theorem 7. $\mathrm{ch}(K_{4*k}) = l(4,k) := \lceil \frac{3k-1}{2} \rceil$.

Some of the recent development of list coloring of complete multipartite graphs has been motivated by paintability, or on-line choosability. Introduced independently by Schauz [15] and Zhu [21] (as on-line list colouring), *paintability* is a coloring game played between two players Alice and Bob on a graph $G = (V,E)$ and a function $f : V \to \mathbb{N}$.

Let V_i denote the vertex set at the start of round i; so $V_1 = V$. At round i, Alice selects a nonempty set of vertices $A_i \subseteq V_i$, and Bob selects an independent set $B_i \subseteq A_i$. Then B_i is deleted from the graph so that $V_{i+1} = V_i \smallsetminus B_i$, and the rounds are continued until $V_n = \emptyset$. Alice's goal is to present some vertex v more than $f(v)$ times, while Bob's goal is to choose every vertex before it has been presented $f(v)+1$ times. We say that G is *on-line f-choosable* if Bob has a strategy such that any vertex $v \in V$ is in at most $f(v)$ sets A_i, and *on-line k-choosable* if G is on-line f-choosable when $f(v) = k$ for all $v \in V$. The *on-line choice number*, denoted $\mathrm{ch}^{\mathrm{ol}}(G)$, is the least k such that G is on-line k-choosable.

This game formulation hides the on-line nature of the problem. Another way of thinking about it is that Alice has secretly assigned lists of colors to all the vertices. At round i she reveals all vertices whose list contains color i, and Bob colors an independent set of them with color i. In this formulation it is clear that $\mathrm{ch}(G) \leq \mathrm{ch}^{\mathrm{ol}}(G)$.

Schauz [15] proved that surprisingly many results on choice number, including Brooks' theorem [3], Thomassen's theorem [16], and the Bondy–Boppana–Siegel kernel lemma carry over to on-line choice number. Until very recently it was not known whether any graphs G satisfy $\mathrm{ch}^{\mathrm{ol}}(G) - \mathrm{ch}(G) \geq 2$. Now Duraj, Gutowski, and Kozik [4] have proved

$$\mathrm{ch}^{\mathrm{ol}}(K_{s*2}) - \mathrm{ch}(K_{s*2}) = \Omega(\log \mathrm{ch}(K_{s*2})).$$

The explicit value of $\mathrm{ch}(K_{4*k})$ provided by Theorem 7 may be useful for establishing larger gaps. In Section 4 we show that $\mathrm{ch}(K_{4*3}) < \mathrm{ch}^{\mathrm{ol}}(K_{4*3})$.

2. Set-up

Fix $s, k \in \mathbb{Z}^+$. Let $G = (V, E) = K_{s*k}$, and let $\mathcal{P}$ be the partition of V into k independent s-sets. Let $l = l(k, s) = \lceil \frac{(s-1)2k-s+2}{s} \rceil$, and consider any l-list assignment L for G. Put $C^* = \bigcup_{x \in V} L(x)$. Let $L\neg\alpha$ be the result of deleting α from every list of L.

We may write $x_1 \ldots x_t$ for the subpart $S = \{x_1, \ldots, x_t\} \subseteq X \in \mathcal{P}$; when we use this notation we implicitly assume the x_i are distinct. Also set $\overline{S} = X \smallsetminus S$. For a set of vertices $S \subseteq V$ let $\mathcal{L}(S) = \{L(x) : x \in S\}$, $L(S) = \bigcap \mathcal{L}(S)$, $W(S) = \bigcup \mathcal{L}(S)$, and $l(S) = |L(S)|$. The operation of replacing the vertices in S by a new vertex v_S with the same neighborhood as S is called *merging*. The new vertex v_S is said to be *merged*; vertices that are not merged are called *original*. When merging a set S we also create a list $L(v_S) = L(S)$.

For a color $\alpha \in C^*$, let $|X, \alpha| = |\{x \in X : \alpha \in L(x)\}|$ be the number of times α appears in the lists of vertices of X, $N_i(X) = \{\alpha \in C^* : |X, \alpha| = i\}$ be the set of colors that appear exactly i times in the lists of vertices in X, $n_i(X) = |N_i(X)|$, $N(X) = N_2(X) \cup N_3(X)$, and $n(X) = |N(X)|$. Let $\sigma_i(X) = \sum\{l(I) : I \subseteq X \wedge |I| = i\}$ and $\mu_i(X) = \max\{l(I) : I \subseteq X \wedge |I| = i\}$.

For a set S and element x we use the notation $S + x = S \cup \{x\}$ and $S - x = S \smallsetminus \{x\}$. The range of a function f is denoted by $\mathrm{ran}(f)$.

The following lemma was proved independently by Kierstead [8], and by Reed and Sudakov [13], [14], and named by Rabern.

Lemma 8 (Small Pot Lemma)**.** *If $\mathrm{ch}(G) > r$ then there exists a r-list assignment L using fewer than $|V(G)|$ colors such that G is not L-colorable.*

If s does not satisfy (1.1) then there is a minimum counterexample k with $\mathrm{ch}(K_{s*k}) > l(s, k)$. By the Small Pot Lemma, this is witnessed by a list assignment L with $|\bigcup\{L(x) : x \in V(G)\}| < |V|$. We always assume L has this property.

Lemma 9. *If G is a minimum counterexample with witness L, then every part X of G satisfies $L(X) = \emptyset$.*

Proof. Otherwise $\alpha \in L(X)$ for some color α and a part X. Color each vertex in X with α, set $G' = G - X$, and put $L' = L\neg\alpha$. Then L' witnesses that $k-1$ is a smaller counterexample, a contradiction. $\square$

By Lemma 9, $n_s(X) = 0$ for each part $X \in \mathcal{P}$. So by the Small Pot Lemma, $|W(X)| = \sum_{i=1}^{s-1} n_i(X) < sk$. Also $\sum_{i=1}^{s-1} i\, n_i(X) = sl$ is the number of occurrences of colors in the lists of vertices of X. Thus

$$\sum_{i=1}^{s-1}(i-1)n_i(X) \geq sl - |W(X)| \geq s(l-k)+1. \tag{2.1}$$

Now we warm-up by giving a short proof extracted from [11] of Theorem 4.

Proof of Theorem 4. Let $s = 3$, $l = l(3,k) = \mathrm{ch}(K_{3*k}) = \lceil \frac{4k-1}{3} \rceil$, and assume G is a counterexample with k minimal. Then $k > 1$. By Lemma 9, $n_3(X) = 0$ for all $X \in \mathcal{P}$. We obtain a contradiction by L-coloring G. First we use the following steps to partition V into sets of vertices that will receive the same color. Then we *merge* each set I into a single vertex v_I, and assign v_I the set of colors in $L(I)$. Finally we apply Hall's Theorem to choose a system of distinct representatives (SDR) for these new lists; this induces an L-coloring of G.

Step 1. Partition $\mathcal{P}$ into a set $\mathcal{R}$ of $l-k$ *reserved* parts and a set $\mathcal{U} = \mathcal{P} \smallsetminus \mathcal{R}$ of $2k-l$ *unreserved* parts.

Step 2. Choose $\mathcal{U}_1 \subseteq \mathcal{U}$ to maximize $\nu = \sum_{X \in \mathcal{U}_1} \mu_2(X)$ subject to the constraint $|\mathcal{U}_1| \leq \mu_2(X)$ for all $X \in \mathcal{U}_1$. Set $u_1 = |\mathcal{U}_1|$. For each $X \in \mathcal{U}_1$ choose a pair $I_X \subseteq X$ with $l(I_X) \geq u_1$ maximum. Put $\mathcal{U}_2 = \mathcal{U} \smallsetminus \mathcal{U}_1$ and $u_2 = |\mathcal{U}_2|$.

The maximality of ν implies

$$\mu_2(X) \leq u_1 \text{ for all } X \in \mathcal{U}_2, \tag{2.2}$$

as otherwise adding X to $\mathcal{U}_1$, and deleting one part $Y \in \mathcal{U}_1$ with $\mu_2(Y) = u_1$, if such a part Y exists, would increase ν.

Step 3. Using (2.1), each part $X \in \mathcal{P}$ satisfies

$$n_2(X) \geq 3(l-k)+1 = 3\left\lceil \frac{k-1}{3} \right\rceil + 1 \geq k-1+1 = k.$$

Form an SDR f for $\{L(v_{I_X}) : X \in \mathcal{U}_1\} \cup \{N(X) : X \in \mathcal{R}\}$ by greedily choosing representatives for the first family and then for the second family. For each $X \in \mathcal{R}$ choose a pair $I_X \subseteq X$ so that $f(X) \in L(I_X)$.

Step 4. For each $X \in \mathcal{U}_1 \cup \mathcal{R}$, merge I_X to a new vertex v_{I_X}, let $z_X \in X \smallsetminus I_X$, and set $X' = \{v_{I_X}, z_X\}$. If $X \in \mathcal{U}_2$, set $X' = X$. This yields a graph G' with parts $\mathcal{P}' = \{X' : X \in \mathcal{P}\}$, and list assignment L.

Next we use Hall's Theorem to prove that $\{L(x) : x \in V(G')\}$ has an SDR. For this it suffices to prove:

$$|S| \leq \left| \bigcup \{L(x) : x \in S\} \right| \text{ for every } S \subseteq V(G'). \tag{2.3}$$

To prove (2.3), let $S \subseteq V(G')$ be arbitrary, and set $W = W(S) := \bigcup\{L(x) : x \in S\}$. We consider several cases in order, always assuming all previous cases fail.

Case 1: There exists $X \in \mathcal{P}$ with $|S \cap X'| = 3$. Then $X' = X \in \mathcal{U}_2$, $u_2 \geq 1$, and $|S| \leq |G'| \leq 2k + u_2$. By (2.2), $u_1 \geq \mu_2(X) \geq \sigma_2(X)/3$. Using inclusion-exclusion, and Lemma 9,

$$\begin{aligned} |W| &\geq |W(X)| = \sigma_1(X) - \sigma_2(X) + \sigma_3(X) \geq 3l - 3u_1 = 3l - 3(2k - l - u_2) \\ &\geq 6(l-k) + 3u_2 \geq (2k-2) + (2 + u_2) \geq 2k + u_2 \geq |S|. \end{aligned}$$

Case 2: There is $X \in \mathcal{U}_2$ with $|S \cap X'| = 2$. Since $u_1 = 2k - l - u_2 < 2k - l$, By (2.2),

$$|W| \geq |W(S \cap X)| \geq 2l - l(S \cap X) \geq 2l - u_1 \geq 2l - (2k - l - u_2) \geq 3l + 1 - 2k \geq 2k \geq |S|.$$

Case 3: There is $X \in \mathcal{U}_1$ with $|S \cap X'| = 2$. As $|S| \leq 2k - u_2 = l + u_1$ and $L(v_{I_X} z_X) = L(X) = \emptyset$,

$$|W| \geq |W(S \cap X')| \geq l(v_{I_X}) + l(z_X) - l(v_{I_X} z_X) \geq u_1 + l \geq |S|.$$

Case 4: S has an original vertex. Then $|S| \leq 2k - u_1 - u_2 = l \leq |W|$.
Case 5: All vertices of S have been merged. Then $|S| \leq |f(S)| \leq |W|$.

□

3. The main theorem

In this section we prove our main result, Theorem 7. The case when k is odd is considerably more technical. Casual or first time readers may wish to avoid these additional details; the proof is organized so that this is possible. In particular, in the even case Step 7(b), Step 11, Lemma 10(b), and Lemma 14 are not involved. Furthermore, only the first conclusion of Lemma 13 that k is odd (in the bad case covered by its hypothesis) is used. Let $b \in \{0, 1\}$ with $b \equiv k \bmod 2$ and $l = l(4, k) = \lceil \frac{3k-1}{2} \rceil$. We often use the partition $k = (2k - l) + (l - k)$ of the integer k, and note that $2k - l = l - k + b$.

Proof of Theorem 7. Our set-up is the same as in the proof of Theorem 4. Let $s = 4$, $l = l(4, k)$, and $G = K_{4*k}$. The theorem is trivial if $k = 1$. Let $k > 1$ be a minimal counterexample, and let L be an l-list assignment for G with $|W(V)| \leq 4k - 1$ and $L(X) = \emptyset$ for all parts $X \in \mathcal{P}$. Again we partition V into sets of vertices that will receive the same color, and then find an SDR for the induced list assignment that in turn induces an L-coloring of G. See Figure 3.1.

Step 1. Reserve notation for a partition $\mathcal{P} = \mathcal{R} \cup \mathcal{U}$ of V with $|\mathcal{R}| = l - k$, $|\mathcal{U}| = 2k - l$, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, and $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$, where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ are to be defined in the following steps.
Step 2. Choose $\mathcal{U}_1 \subseteq \mathcal{P}$ so that $|\mathcal{U}_1| \leq 2k - l$, for every $X \in \mathcal{U}_1$ there is a pair $I_X \subseteq X$ with (*) $l(I_X), l(\overline{I}_X) \geq k$, and subject to these constraints $|\mathcal{U}_1|$ is maximum. For each $X \in \mathcal{U}_1$ fix I_X witnessing (*). Let $u_1 := |\mathcal{U}_1|$. Then:

$$\text{(3.1)} \quad \text{If } u_1 < 2k - l \text{ then } (\forall X \in \mathcal{P} \setminus \mathcal{U}_1)(\forall I \subseteq X)[|I| = 2 \to \min\{l(I), l(\overline{I})\} \leq k - 1].$$

Step 3. Choose $\mathcal{U}_2 \subseteq \mathcal{P} \setminus \mathcal{U}_1$ so that $|\mathcal{U}_2| \leq 2k - l - u_1$ and $|\mathcal{U}_2| \leq \mu_3(X)$ for all $X \in \mathcal{U}_2$; subject to this let $\nu_3 = \sum_{X \in \mathcal{U}_2} \mu_3(X)$ be maximum. Let $u_2 = |\mathcal{U}_2|$. If $\mathcal{U}_2 \neq \emptyset$ we select a part $\dot{Z} \in \mathcal{U}_2$; else $\dot{Z} = \emptyset$. For each $X \in \mathcal{U}_2$ choose a triple $I_X \subseteq X$ with $l(I_X) \geq u_2$ maximum. Since ν_3 cannot be increased:

$$\text{(3.2)} \qquad \text{If } u_1 + u_2 < 2k - l \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R})[\mu_3(X) \leq u_2].$$

Step 4. Choose $\mathcal{R}_1 \subseteq \mathcal{P} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$ so that $|\mathcal{R}_1| \leq l - k$ and there is a family of sets $\{I_X : X \in \mathcal{R}_1\}$ such that (*) $I_X \subseteq X$, $|I_X| = 3$, and there is an SDR f_1 of $\mathcal{L}(M_1)$, where

$M_1 := \{v_{I_X} : X \in \mathcal{U}_2 \cup \mathcal{R}_1\}$; subject to this constraint, choose $\mathcal{R}_1$ with $|\mathcal{R}_1|$ maximum. Fix $\{I_X : X \in \mathcal{R}_1\}$, f_1 and M_1 satisfying (*). Let $C_1 = \mathrm{ran}(f_1)$ and $r_1 := |\mathcal{R}_1|$. Then:

$$\text{If } r_1 < l-k \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[N_3(X) \subseteq C_1]. \tag{3.3}$$

Moreover, by Lemma 9, $L(T) \cap L(T') = \emptyset$ for any two triples $T, T' \subseteq X$, and so

$$\text{If } r_1 < l-k \text{ then } (\forall X \in \mathcal{U}_3 \cup \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[\sigma_3(X) \leq u_2 + r_1]. \tag{3.4}$$

Step 5. Choose $\mathcal{U}_3 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{R}_1)$ so that $|\mathcal{U}_3| \leq 2k-l-u_1-u_2$ and $l-k+u_2+|\mathcal{U}_3| \leq \mu_2(X)$ for all $X \in \mathcal{U}_3$; subject to this constraint let $\nu_5 = \sum_{X \in \mathcal{U}_3} \mu_2(X)$ be maximum. Let $u_3 = |\mathcal{U}_3|$. Since ν_5 cannot be increased:

$$\text{If } u_1 + u_2 + u_3 < 2k - l \text{ then } (\forall X \in \mathcal{U}_4 \cup \mathcal{R}_2 \cup \mathcal{R}_3)[\mu_2(X) \leq l - k + u_2 + u_3]. \tag{3.5}$$

For all $X \in \mathcal{U}_3$ choose a pair $I_X = xy \subseteq X$ with $l(I_X) \geq l - k + u_2 + u_3$ maximum; subject to this choose I_X so that $\Delta_1(I_X) := l(I_X) - l(\overline{I}_X)$ is maximum. Set $\Delta_2(I_X) = 2u_2 - l(xyz) - l(xyw)$, where $zw = \overline{I}_X$.
Step 6. Choose $\mathcal{R}_2 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1)$ so that $|\mathcal{R}_2| \leq l - k - r_1$ and $\sigma_2(X) - \sigma_3(X) \geq 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + |\mathcal{R}_2|$ for all $X \in \mathcal{R}_2$; subject to this constraint let $\nu_6 = \sum_{X \in \mathcal{R}_2}(\sigma_2(X) - \sigma_3(X))$ be maximum. Set $r_2 = |\mathcal{R}_2|$. Then:

$$\begin{gathered}\text{If } r_1 + r_2 < l - k \text{ then } (\forall X \in \mathcal{U}_4 \cup \mathcal{R}_3) \\ [\sigma_2(X) - \sigma_3(X) \leq 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2].\end{gathered} \tag{3.6}$$

Step 7. Choose $\mathcal{R}_3 \subseteq \mathcal{P} \smallsetminus (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1 \cup \mathcal{R}_2)$ with $|\mathcal{R}_3| = l - k - r_1 - r_2$. Let $r_3 = |\mathcal{R}_3|$. If $r_3 = 0$ then set $M_2 = M_1 \cup \{v_{I_X} : X \in \mathcal{U}_3\}$, greedily extend f_1 to an SDR f_3 of $\mathcal{L}(M_2)$, and go to Step 8.

Otherwise, $r_3 \geq 1$. For $I \subseteq X \in \mathcal{R}_3$, put $L^1(I) = L(I) \smallsetminus C_1$, $l^1(I) = |L^1(I)|$, and if $|I| = 2$, let $\Delta_1^1(I) = l^1(I) - l^1(\overline{I})$. By Lemma 10, all 2-sets $I \subset X$ satisfy

$$W(\overline{I} + v_I) \geq 3k - b + \Delta_1^1(I) - u_2 - r_1. \tag{3.7}$$

Set $\dot{r} = 0$ and do one of (a) or (b) below.

(a) If $b = 0$ or there are both $X \in \mathcal{R}_3$ and a 2-set $I \subset X$ with $\Delta_1^1(I) \geq 1$, then choose $\dot{Y} \in \mathcal{R}_3$ and a 2-set $I_{\dot{Y}} \subset \dot{Y}$ so that $\Delta_1^1(I_{\dot{Y}})$ is maximum and $L(I_{\dot{Y}}) \neq \emptyset$. By Lemma 10(a) there is a family $\mathcal{I}$ with $I_{\dot{Y}} \in \mathcal{I} = \{I_X : X \in \mathcal{R}_3\}$ such that $I_X \subseteq X$, $|I_X| = 2$, and $\Delta_1^1(I_X) \geq 0$ for all $X \in \mathcal{R}_3$; and $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\})$ has an SDR f_2 extending f_1. Fix such $\mathcal{I}$ and f_2, set $M_2 = M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\}$, and set $C_2 = \mathrm{ran}(f_2)$.

(b) Else $b = 1$, and $\Delta_1^1(I) = 0$ for all $X \in \mathcal{R}_3$ and all 2-sets $I \subset X$. Pick any class $\dot{Y} \in \mathcal{R}_3$. By Lemma 10(b), there is a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_3\}$ such that $I_X \subseteq X$, $|I_X| = 2$, and $\Delta_1^1(I_X) \geq 0$ for all $X \in \mathcal{R}_3$; and $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\} + v_{\overline{I}_{\dot{Y}}})$ has an SDR h extending f_1. Let $M_2 = M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\}$, f_2 equal h restricted to M_2, and $C_2 = \mathrm{ran}(f_2)$. (b*) If $u_1 = 0 = r_2$ then Step 9 is degenerate; in this case set $f_3 = h$, $M_3 = M_2 + v_{\overline{I}_{\dot{Y}}}$, $C_3 = \mathrm{ran}(f_3)$, and $\dot{r} = 1$.
Step 8. Put $\mathcal{U}_4 := \mathcal{U} \smallsetminus (\mathcal{R} \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3)$, and $u_4 := |\mathcal{U}_4|$.
Step 9. If $u_1 = 0 = r_2$ then go to Step 10. Otherwise, using Lemma 12, choose a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_2\}$ such that $I_X \subseteq X$ and $|I_X| = 2$ for all $X \in \mathcal{R}_2$, and $\mathcal{L}(M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$ has an SDR f_3 that extends f_2. Set $M_3 = M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$ and $C_3 = \mathrm{ran}(f_3)$.
Step 10. Let $G' := (V', E')$ be the graph obtained from G by merging each I_X with $X \in \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ and each $\overline{I}_X$ with $X \in \mathcal{U}_1 \cup \mathcal{R}_2$. Note that this does not include $\overline{I}_{\dot{Y}}$ even if Step 7(b*) is executed. For a part X, let X' be the corresponding part in G', and set $\mathcal{P}' = \{X' : X \in \mathcal{P}\}$.

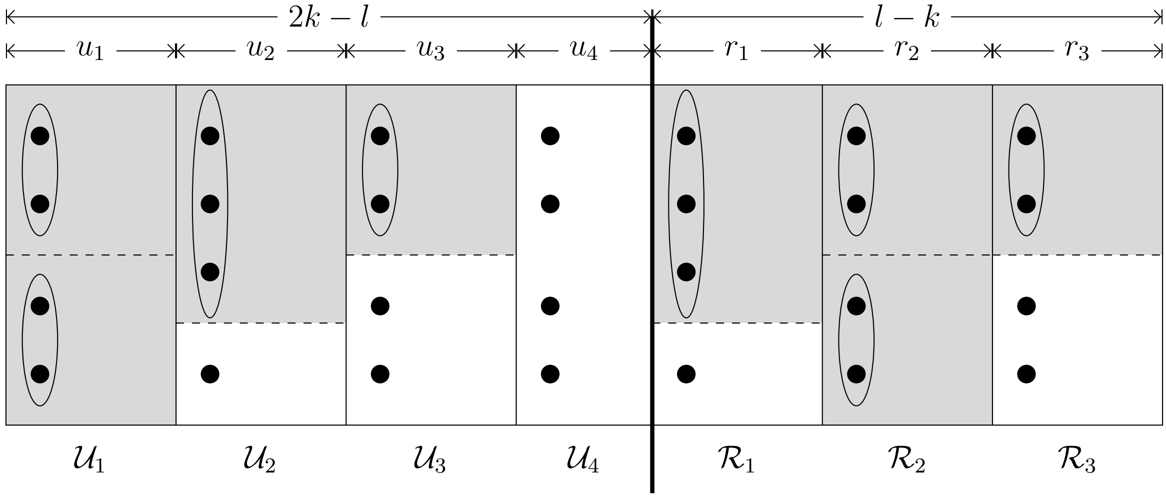

FIGURE 3.1. The partition $\mathcal{P}$ of K_{4*k}: The ovals indicate sets of vertices that are merged to form $\mathcal{P}'$.

Step 11. Set $\dot{u} = \ddot{u} = 0$. If k is odd ($b = 1$) then we merge one more pair of vertices under the following special circumstances:

(a) There exists $X \in \mathcal{U}_4$ with $|W(X)| < |G'|$. Fix such an $X = \dot{X}$. By Lemma 13, $u_1 = 0 = r_3$ and there is a pair $I_{\dot{X}} \subseteq \dot{X}$ such that (i) f_3 can be extended to an SDR f of $\mathcal{L}(M)$, where $M := M_3 + v_{I_{\dot{X}}}$; (ii) $|W(\{v_{I_{\dot{X}}}, v\})| \geq 2k - 1$, and if equality holds then $|W(\{v_{I_{\dot{X}}}, v\} \cup \dot{Z}') \cup C_3| \geq 2k$ for both $v \in \overline{I}_{\dot{X}}$; and (iii) $W(\overline{I}_{\dot{X}} + v_{I_{\dot{X}}}) \geq |G'| - 1$. Merge $I_{\dot{X}}$ and set $\dot{u} = 1$.

(b) Condition (a) fails and there exist $X \in \mathcal{U}_4$ and $xyz \subseteq X$ with

$$|W(xyz \cup \dot{Z}')| \leq 2k + u_4 - 1 < |W(X)|.$$

Fix such an $X = xyzw = \ddot{X}$. By Lemma 14 there is a pair $I_{\ddot{X}} \subseteq xyz$ such that (i) f_3 can be extended to an SDR f of $\mathcal{L}(M)$, where $M := M_3 + v_{I_{\ddot{X}}}$; (ii) $|W(\{v_{I_{\ddot{X}}}, v\})| \geq 2k$ for $v \in xyz \smallsetminus I_{\ddot{X}}$ and $|W(\{v_{I_{\ddot{X}}}, w\})| \geq 2k - 1 + u_3$; and (iii) $|W(\overline{I}_{\ddot{X}} + v_{I_{\ddot{X}}})| \geq 2k + u_4$. Merge $I_{\ddot{X}}$ and set $\ddot{u} = 1$.

Step 12. Recall that G' is the graph obtained after the first ten steps. Let H be the final graph obtained by this merging procedure, including $\overline{I}_{\dot{Y}}$ if Step 7(b*) is executed. (If $b = 0$, and possibly otherwise, $H = G'$). Also let M be the final set of merged vertices, f be the final SDR of $\mathcal{L}(M)$, and $C = \mathrm{ran}(f)$.

Recall that f_i is an SDR for $\mathcal{L}(M_i)$ with $\mathrm{ran}(f_i) = C_i$, M is the final set of merged vertices, f is the final SDR of $\mathcal{L}(M)$, and $C = \mathrm{ran}(f)$. Also, depending on whether Step 7(b*) is executed, $M_3 = M_2 + v_{I_{\dot{Y}}}$ or $M_3 = M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$. The following table summarizes some of the notation from the algorithm before Step 11.

i	$X \in \mathcal{U}_i$	$X \in \mathcal{U}_i$	$X \in \mathcal{R}_i$	M_i
1	$\lvert I_X\rvert = 2$	$l(I_X), l(\overline{I}_X) \geq k$	$\lvert I_X\rvert = 3$	$\{v_{I_X} : X \in \mathcal{U}_2 \cup \mathcal{R}_1\}$
2	$\lvert I_X\rvert = 3$	$l(I_X) \geq u_2$	$\lvert I_X\rvert = 2$	$M_1 \cup \{v_{I_X} : X \in \mathcal{U}_3 \cup \mathcal{R}_3\}$
3	$\lvert I_X\rvert = 2$	$l(I_X) \geq l - k + u_2 + u_3$	$\lvert I_X\rvert = 2$	$M_2 \cup \{v_{I_X}, v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\}$ or $M_2 + v_{\overline{I}_{\dot{Y}}}$

TABLE 1. Notation and facts from the algorithm

Our next task is to state and prove the four lemmas on which the algorithm is based. The first lemma is used for Step 7. The statement of the lemma uses the notation from that step.

Lemma 10. *Every* 2*-set* $I \subset X \in \mathcal{R}_3$ *satisfies*

$$W(\overline{I} + v_I) \geq 3k - b + \Delta_1^1(I) - u_2 - r_1. \tag{3.8}$$

Furthermore, for all $Y \in \mathcal{R}_3$*:*

(a) There is a 2*-set* $I_0 \subset Y$ *with* $L^1(I_0) \neq \emptyset$ *and* $\Delta_1^1(I_0) \geq 0$*. For every such* I_0 *there is a family* $\mathcal{I} = \{I_X : X \in \mathcal{R}_3\}$ *with* $I_Y = I_0$ *such that* $I_X \subseteq X$*,* $|I_X| = 2$*, and* $\Delta_1^1(I_X) \geq 0$ *for all* $X \in \mathcal{R}_3$*; and* $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\})$ *has an SDR* f_2 *extending* f_1*.*

(b) Additionally, if (H) $\Delta_1^1(I) = 0$ *for all* $X \in \mathcal{R}_3$ *and all* 2*-sets* $I \subset X$*, then for some* 2*-set* $I_Y \subset Y$*, the family* $\mathcal{I}$ *and SDR* f_2 *from part (a) can be chosen so that there is an SDR* f_3 *of* $\mathcal{L}(M_1 \cup \{v_{I_X} : X \in \mathcal{R}_3 \cup \mathcal{U}_3\} + v_{\overline{I}_Y})$ *extending* f_2*.*

Proof. Consider any $X = xyzw \in \mathcal{R}_3$. For $I \subseteq X$, let $l^2(I) = |L(I) \cap C_1|$. Then $l(I) = l^1(I) + l^2(I)$. First we prove (3.8) with $I = xy$. By (3.3), $l^1(wxy) = 0 = l^1(zxy)$; using Lemma 9, $l(X) = 0$ and $l^2(xy) - l^2(wxy) - l^2(zxy) \geq 0$; and $l^2(wz) \leq |C_1| = u_2 + r_1$. Thus

$$\begin{aligned} |W(wz + v_{xy})| &= l^1(w) + l^1(z) + l^1(xy) - l^1(wz) - l^1(wxy) - l^1(zxy) + \\ &\quad l^2(w) + l^2(z) + l^2(xy) - l^2(wz) - l^2(wxy) - l^2(zxy) \\ &\geq 2l + \Delta_1^1(xy) - |C_1| \geq 3k - b + \Delta_1^1(xy) - u_2 - r_1. \end{aligned}$$

Let $A(X) = N(X) \setminus C_1$. Then $A(X)$ is the set of colors available for coloring a merged pair of vertices from X. By (3.3), $N_3(X) \subseteq C_1$. Thus $A(X) \subseteq N_2(X)$ and $L^1(I) = L(I) \setminus C_1 = L(I) \cap A(X)$ for all pairs $I \subseteq X$. By Lemma 9, $\{L^1(I) : I \subseteq X \wedge |I| = 2\}$ is a partition of $A(X)$. For each color $\alpha \in A(x)$, set $I(\alpha) = \{x \in X : \alpha \in L(x)\}$. As $\alpha \in A(X) \subseteq N_2(X)$, $|I(\alpha)| = 2$. Let $B(X) = \{\alpha \in A(X) : \Delta_1^1(I(\alpha)) \geq 0)\}$. Colors $\beta \in B(X)$, do not conflict with f_1, can be representatives for $L(I(\beta))$, and satisfy $\Delta_1^1(I(\beta)) \geq 0$.

Setting $X = wx_1x_2x_3$, and using $N_3(X) \subseteq C_1$,

$$|A(X)| = \sum_{i=1}^{3}(l^1(wx_i) + l^1(\overline{wx}_i)) \leq 2\sum_{i=1}^{3}\max(l^1(wx_i), l^1(\overline{wx}_i)) \leq 2|B(X)|. \tag{3.9}$$

By (2.1)

$$n_2(X) + 2n_3(X) \geq 4l - |W(X)| \geq 4(l - k) + 1 \geq 2k - 1. \tag{3.10}$$

As $N_3(X) \subseteq C_1$, $n_3(X) \leq |C_1| = u_2 + r_1$, and

$$\begin{aligned} |A(X)| &\geq n_2(X) + n_3(X) - |C_1| \geq n_2(X) + 2n_3(X) - n_3(X) - |C_1| \\ &\geq 2k - 1 - (2u_2 + 2r_1) \geq 2r_3 + 2u_3 - 1. \end{aligned} \tag{3.11}$$

(3.9), $|B(X)| \geq \lceil |A(X)|/2 \rceil \geq r_3 + u_3$. Thus the first sentence of (a) holds.

For the rest of (a), we construct the family $\mathcal{I}$ and SDR f_2 by a greedy algorithm. Start with the special class $Y \in \mathcal{R}_3$, and its preassigned subset I_Y. As $L^1(I_Y) \neq \emptyset$, and $\Delta_1^1(I_Y) \geq 0$ there is a color $\alpha \in L^1(I_Y) \cap B(Y)$. Let $f_2(L(v_{I_Y})) = \alpha$. Next process all $X \in \mathcal{R}_3 - Y$ one at a time. When X is considered, at most $r_3 - 1$ of the $r_3 + u_3$ colors of $B(X)$ have been used. Let β be an unused color, set $I_X = I(\beta)$, and put $f_2(L(v_{I_X})) = \beta$. Finally consider all $Z \in \mathcal{U}_3$. Recall that I_Z has been assigned in Step 3 so that $l(I_Z) \geq l - k + u_2 + u_3 \geq u_2 + r_1 + u_3 + r_3$. So there is a color $\gamma \in L(I_Z) \setminus C_1$ that has not been used for any previous choices. Set $f_2(L(v_{I_Z})) = \gamma$.

For (b), suppose (H) holds. Thus $|A(X)|$ is even and $A(X) = B(X)$ for all $X \in \mathcal{R}_3$. Again we use a greedy procedure. First choose representatives for each $L(v_{I_Z})$ with $Z \in \mathcal{U}_3$. Also, for each representative α of $L(v_{I_Z}), Z \in \mathcal{U}_3$, remove α from all lists $L^1(I), I \subset X \in \mathcal{R}_3$; and for bookkeeping also remove some additional colors so that for the new lists $L^-(I), I \subset xyzw := X$ satisfy

$$\begin{aligned} |L^-(xy)| &= |L^-(wz)|,\ |L^-(xz)| = |L^-(wy)|,\ |L^-(xw)| = |L^-(yz)|, \text{ and} \\ r_3 &= |L^-(wx)| + |L^-(wy)| + |L^-(wz)|. \end{aligned}$$

Finish the construction by first choosing a 2-set $I_Y \subset Y$ with $L^-(I_Y) \neq \emptyset$, and setting $f_2(L(v_{I_Y})) = \alpha \in L^-(I_Y)$ and $f_2(L(v_{\overline{I}_Y})) = \beta \in L^-(\overline{I}_Y)$. Then for each $X \in \mathcal{R}_3 - Y$, greedily choose a 2-set $I_X \subset X$ so that $L^-(I_X)$ has an unused color γ and set $f_2(L(v_{I_X})) = \gamma$. This is possible since $\sum_{I \subset X, |I|=2} |L^-(I)| = 2r_3$. □

The next lemma is used in Step 9. The statement of the lemma uses the notation from that step. We will need the following easy claim.

Claim 11. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be the three partitions of a 4-set X into pairs. For all $I_1 \in \mathcal{P}_1, I_2 \in \mathcal{P}_2, I_3 \in \mathcal{P}_3$ there exists $v \in X$ such that either (i) $v \in I_1 \cap I_2 \cap I_3$ or (ii) $v \notin I_1 \cup I_2 \cup I_3$.

Lemma 12. *There is a family $\mathcal{I} = \{I_X : X \in \mathcal{R}_2\}$ such that $I_X \subseteq X$ and $|I_X| = 2$ for all $X \in \mathcal{R}_2$, and $\mathcal{L}(M_2 \cup \{v_{I_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\} \cup \{v_{\overline{I}_X} : X \in \mathcal{U}_1 \cup \mathcal{R}_2\})$ has an SDR f_3 that extends f_2.*

Proof. If $r_2 = 0 = u_1$ then there is nothing to prove; else Step 7(b*) is not executed. Each $X \in \mathcal{U}_1$ satisfies $l(I_X), l(\overline{I}_X) \geq k$ by Step 2 and $L(I_X) \cap L(I_{\overline{X}}) = \emptyset$ by Lemma 9. Thus $|L(I_X) \smallsetminus C_2|, |L(\overline{I}_X) \smallsetminus C_2| \geq k - u_2 - u_3 - r_1 - r_3 \geq u_1$. By Theorem 3, $\{L(I_X) \smallsetminus C_2, L(\overline{I}_X) \smallsetminus C_2 : X \in \mathcal{U}_1\}$ has an SDR, and so f_2 can be extended to an SDR g for $\mathcal{L}(M_2')$, where $M_2' := M_2 \cup \{I_X, \overline{I}_X : X \in \mathcal{U}_1\}$. Let $C^g = \operatorname{ran}(g)$. Then

$$|C^g| = 2u_1 + u_2 + u_3 + r_1 + r_3. \tag{3.12}$$

Next consider any $X \in \mathcal{R}_2$. Let $A(X) = N_2(X) \smallsetminus C^g$. Again by Theorem 3 it suffices to show:

$$(\exists I_X \subseteq X)[|I_X| = 2 \wedge |L(I_X) \cap A(X)| \geq r_2 \wedge |L(\overline{I}_X) \cap A(X)| \geq r_2]. \tag{3.13}$$

Observe $\sigma_2(X) = n_2(X) + 3n_3(X)$ and $\sigma_3(X) = n_3(X)$. So $n(X) = n_2(X) + n_3(X) = \sigma_2(X) - 2\sigma_3(X)$. By (3.3), $N_3(X) \subseteq C^g$ and by (3.4) $\sigma_3(X) \leq u_2 + r_1$. So by the choice of X in Step 6,

$$\begin{aligned} n(X) &= \sigma_2(X) - 2\sigma_3(X) \geq 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2 - (u_2 + r_1) \\ &\geq 5(l-k) + 2u_1 + u_2 + u_3 + r_2, \end{aligned} \tag{3.14}$$

and by (3.3),

$$\begin{aligned} |A(X)| &= |N_2(X) \smallsetminus C^g| = |(N_2(X) \cup N_3(X)) \smallsetminus C^g| \geq n(X) - |C^g| \\ &\geq 5(l-k) + 2u_1 + u_2 + u_3 + r_2 - (2u_1 + u_2 + u_3 + r_1 + r_3) \\ &\geq 5(l-k) - r_1 + r_2 - r_3 \geq 4(l-k) + 2r_2. \end{aligned} \tag{3.15}$$

Suppose (3.13) fails. Then for each of the three partitions of X into pairs, there is a pair uv with $|L(uv) \cap A(X)| \leq r_2 - 1$. Using Claim 11, there exists $w \in X$ such that either (i) $|L(vw) \cap A(X)| \le r_2 - 1$ for all $v \in X - w$ or (ii) $|L(uv) \cap A(X)| \le r_2 - 1$ for all $u, v \in X - w$. For all $v \in X$,

$$\text{(3.16)} \qquad \text{(a) } |L(v) \cap N(X)| \le l \text{ and (b) } |L(v) \cap N(X)| \le \sum_{u \in X - v} l(uv).$$

If (i) holds then

$$\text{(3.17)} \qquad |L(w) \cap N(X)| \le |C^g| + \sum_{v \in X - w} |L(vw) \cap A(X)| \le |C^g| + 3r_2 - 3.$$

Using (3.17) and (3.16a),

$$\text{(3.18)} \qquad 2n(X) \le |L(w) \cap N(X)| + \sum_{v \in X - w} |L(v) \cap N(X)| \le (|C^g| + 3r_2 - 3) + 3l.$$

By (3.14), (3.18) and (3.12) we have

$$\text{(3.19)} \qquad \begin{aligned} 10(l-k) + 4u_1 + 2u_2 + 2u_3 + 2r_2 &\le 3l + 2u_1 + u_2 + u_3 + r_1 + r_3 + 3r_2 - 3 \\ (6l - 9k + 3) + 2u_1 + u_2 + u_3 &\le k - l + r_1 + r_2 + r_3 = 0. \end{aligned}$$

Since $6l - 9k = -3b$, both $b = 1$ and $0 = u_1 = u_2 = u_3$. By (3.5),

$$\mu_2(X) \le l - k + u_2 + u_3 = l - k.$$

Using this with (3.16b) in (3.18) to strengthen the estimate in (3.19) gives a contradiction:

$$\begin{aligned} 10(l-k) + 2r_2 &\le 9(l-k) + (|C^g| + 3r_2 - 3) \\ l - k &\le r_1 + r_2 + r_3 - 3 < l - k. \end{aligned}$$

Thus (ii) holds. So

$$\text{(3.20)} \qquad |A(X)| \le l(w) + \sum_{uv \subseteq X - w} |L(uv) \cap A(X)| \le l + 3(r_2 - 1).$$

Using (3.15) and (3.20),

$$\begin{aligned} 4(l-k) + 2r_2 &\le |A(X)| \le l + 3(r_2 - 1) \\ 3l - 4k + 3 &\le r_2. \end{aligned}$$

As $2l - 3k = -b$, this yields the contradiction $l - k + 2 \le r_2 \le l - k$. □

The next Lemma is used in Step 11(a). Recall that $\dot{Z}$ is defined in Step 3, and Δ_1 and Δ_2 are defined in Step 5.

Lemma 13. *Suppose $X = xyzw \in \mathcal{U}_4$ and $|W(X)| < |G'|$. Then $b = 1$, $u_1 = 0 = r_3$, $u_2 + u_3 \ge 1$, and there exists a pair $J \subseteq X$ such that:*

(1) $L(J) \nsubseteq C_3$, and so f_3 can be extended to an SDR of $\mathcal{L}(M_3 + v_J)$;
(2) for both $v \in \overline{J}$, both $|W(\{v_J, v\})| \ge 2k - 1$ and if $|W(\{v_J, v\})| = 2k - 1$ then $|W(\{v_J, v\} \cup \dot{Z}') \cup C_3| \ge 2k$;
(3) $|W(\overline{J} + v_J)| \ge |G'| - 1$; in particular $|W(X)| \ge |G'| - 1$.

Proof. Now $|G'| = 3k - u_1 - u_2 + u_4 - r_1 - r_2$. Observe that

$$\text{(3.21)} \qquad \sigma_2(X) - \sigma_3(X) \ge 5(l-k) + 2u_1 + 2u_2 + u_3 + r_1 + r_2 + 1,$$

since otherwise inclusion-exclusion yields the contradiction:

$$\begin{aligned}|W(X)| &= \sigma_1(X)-\sigma_2(X)+\sigma_3(X)\\ &\geq 4l-5(l-k)-2u_1-2u_2-u_3-r_1-r_2\\ &= 3k+(2k-l-u_1-u_2-u_3)-u_1-u_2-r_1-r_2\\ &= 3k+u_4-u_1-u_2-r_1-r_2=|G'|>|W(X)|.\end{aligned}$$

By (3.21) and (3.6), $r_1+r_2=l-k$ and $r_3=0$. Consider any pair $I=xy\subseteq X$. Then

$$\begin{aligned}|W(\overline{I}+v_I)| &\geq l(z)+l(w)+l(xy)-l(xyz)-l(xyw)-l(zw)\\ &= 2l+\Delta_1(I)-2u_2+\Delta_2(I)\end{aligned}$$

$$\begin{aligned}1\leq |G'|-|W(\overline{I}+v_I)| &\leq b-2u_1+(u_1+u_2+u_4-l+k)-\Delta_1(I)-\Delta_2(I)\\ &= 2b-2u_1-u_3-\Delta_1(I)-\Delta_2(I); \qquad (3.22)\end{aligned}$$

$$\Delta_1(I)+\Delta_2(I)\leq 2b-1-2u_1-u_3, \qquad (3.23)$$

and similarly for the pair zw,

$$\Delta_1(\overline{I})+\Delta_2(\overline{I})\leq 2b-1-2u_1-u_3. \qquad (3.24)$$

As $\Delta_1(I)=-\Delta_1(\overline{I})$, and $\Delta_2(I),\Delta_2(\overline{I})\geq 0$ by (3.2), either $\Delta_1(I)+\Delta_2(I)\geq 0$ or $\Delta_1(\overline{I})+\Delta_2(\overline{I})\geq 0$. So $b=1$, $u_1=0$, $u_3\leq 1$, and

$$0\leq 4u_2-\sigma_3(X)=\Delta_2(I)+\Delta_2(\overline{I})=\Delta_1(I)+\Delta_2(I)+\Delta_1(\overline{I})+\Delta_2(\overline{I})\leq 2. \qquad (3.25)$$

By (3.21), $r_1+r_2=l-k$, $\sigma_2(X)\leq 6\mu_2(X)$, (3.5), and $\sigma_3=4u_2-\Delta_2(I)-\Delta_2(\overline{I})$,

$$\begin{aligned}1+6(l-k)+2u_2+u_3+\sigma_3(X) &\leq \sigma_2(X)\leq 6(l-k+u_2+u_3) \qquad (3.26)\\ 1+u_3+6(l-k+u_2)-\Delta_2(I)-\Delta_2(\overline{I}) &\leq \sigma_2(X)\leq 6(l-k+u_2+u_3). \qquad (3.27)\end{aligned}$$

By (3.26) $u_2+u_3\geq 1$. So the first three assertions of the lemma have been proved. It remains to find a pair $J\subseteq X$ satisfying (1–3).

First suppose $u_3=1$. By (3.23), $\Delta_1(I)+\Delta_2(I)=0$ for all pairs $I\subseteq X$. So $\Delta_1(I)\leq 0$ and $\Delta_1(\overline{I})\leq 0$. As $\Delta_1(I)=-\Delta_1(\overline{I})$, this implies $\Delta_1(I)=0=\Delta_1(\overline{I})$. Thus $\Delta_2(I)=0=\Delta_2(\overline{I})$. By (3.27), there exists a pair $I\subseteq X$ with $l(I)\geq l-k+u_2+u_3$. As $\Delta_1(I)=0$, $l(\overline{I})\geq l-k+u_2+u_3$. Thus, using Lemma 9 and (3.2),

$$|W(\{v_I,v_{\overline{I}}\})|=l(I)+l(\overline{I})\geq 2(l-k+u_2+u_3)>2(l-k)+u_2+u_3\geq |C_3|.$$

Pick $J\in\{I,\overline{I}\}$ such that $L(J)\nsubseteq C_3$. Then (1) holds. For (2), let $v'\in\overline{J}$. Using (3.2),

$$|W(\{v_J,v'\})|=l(J)+l(v')-l(J+v')\geq 2l-k+u_2+u_3-u_2\geq 2k.$$

Thus (2) holds. As $u_3=1$, (3.22) implies (3).

Otherwise $u_3=0$. Then $u_2\geq 1$, and so $\dot{Z}\neq\emptyset$. Put $C_0:=C_3\cup W(\dot{Z}')$. By Step 3 and Lemma 9, $|C_0|\geq |W(\dot{Z}')|\geq l+u_2$.

Call a pair $J\subseteq X$ *bad* if $L(J)\subseteq C_3$; otherwise J is *good*. A good pair satisfies (1). Call a vertex $v\in X$ *bad* if $|L(v)\cup C_0|\leq 2k-1$; otherwise v is *good*. If $v\in\overline{J}\subset X$, where $|J|=2$, and v is good, then $|W(\{v_J,v\}\cup\dot{Z}')\cup C_3|\geq 2k$ as in (2).

By (3.2), every triple $T\subseteq X$ satisfies $l(T)\leq u_2$. If equality holds then call T *normal*; otherwise call T *abnormal*; if $l(T)\leq u_2-2$ then call T *very abnormal*. Using (3.25), (3.27) and $u_3=0$ yields

$$6(l-k+u_2)-1\leq\sigma_2\leq 6(l-k+u_2), \qquad (3.28)$$

so by (3.5), every pair $I \subseteq X$ satisfies

$$l-k+u_2-1 \le l(I) \le l-k+u_2.$$

If $l(I)=l-k+u_2$ then call I *normal*; otherwise call I *abnormal*.

Suppose $J \subset T \subset X$ for a normal pair J and an abnormal triple T. As $l(J)$ is maximum, $\Delta_1(J) \ge 0$ and $\Delta_2(J) \ge 1$. By (3.22), J satisfies (3). Finally, both $v \in \overline{J}$ satisfy,

$$|W(\{v_J, v\})| = l(J)+l(v)-l(J+v) \ge \begin{cases} 2l-k+u_2-(u_2-1)=2k & \text{if } v \in T \smallsetminus J \\ 2l-k+u_2-u_2=2k-1 & \text{if } v \in X \smallsetminus T \end{cases}.$$

We have proved the following observation.

Observation 1. If $J \subseteq T \subseteq X$ and $w \in X \smallsetminus T$, where J is a good, normal pair and T is an abnormal triple, then J satisfies (1–3) provided w is good or $C_0 \nsubseteq W(v_J, w)$.

The case $u_3=0$, (3.27), and (3.25) imply $1 \le \Delta_2(I)+\Delta_2(\overline{I}) \le 2$. Thus $\sigma_3(X)=4u_2-2$ or $\sigma_3(X)=4u_2-1$ since $\sigma_3=4u_2-\Delta_2(I)-\Delta_2(\overline{I})$. In the first case either there are two abnormal triples or there is one very abnormal triple; moreover there is at most one abnormal pair by (3.27). In the second case, there is one abnormal triple, and since equality holds in (3.27), there are no abnormal pairs.

Suppose $T=xyz$ is a very abnormal triple. As there is at most one abnormal pair, assume xz and yz are both normal. Then $l(T)=u_2-2$, $l(xz)=l-k+u_2=l(yz)$, and

$$\begin{aligned} |L(z)-W(xy)| &\le l-(2(l-k+u_2)-u_2+2)=2k-l-u_2-2=u_4-2, \text{ so} \\ l(wz) &\le l(xzw)+l(yzw)+|L(z)-W(xy)| \le 2u_2+u_4-2=l-k+u_2-1. \end{aligned}$$

Thus wz is abnormal, and all pairs of T are normal. Now letting x play the role of z, shows that wx is abnormal, a contradiction. So there are no very abnormal triples.

If $x \in X$ is bad then $|C_0 \smallsetminus L(x)| \le 2k-1-l=l-k$. If $y \in X-x$ is also bad, then using (3.5) and $b=1$,

$$\begin{aligned} l-k+u_2 \ge l(xy) &\ge |L(xy)\cap C_0| \ge |C_0|-|C_0 \smallsetminus L(x)|-|C_0 \smallsetminus L(y)| \\ &\ge l+u_2-2(l-k) \ge l-k+u_2+1, \end{aligned}$$

a contradiction. So at most one vertex of X is bad.

Suppose two pairs $I_1, I_2 \subset T \subset X$ are both bad. At least one, say I_1, is normal. As $u_1=0=u_3$,

$$\begin{aligned} (3.29) \qquad 2(l-k)+u_2 &\ge |C_3| \ge |L(I_1)\cup L(I_2)| \ge l-k+u_2+l(I_2)-l(I_1\cup I_2) \\ l(T)=l(I_1\cup I_2) &\ge l(I_2)-l+k = \begin{cases} u_2 & \text{if } I_2 \text{ is normal} \\ u_2-1 & \text{if } I_2 \text{ is abnormal} \end{cases}. \end{aligned}$$

So an abnormal triple contains at most one bad, normal pair.

Suppose there are two abnormal triples. Choose an abnormal triple T so that if there is a bad vertex then it is in T. As T contains three pairs, of which at most one is abnormal, and at most one is bad and normal, T contains a good, normal pair J. Say $J=yz$, $T=xyz$, and $w \in X \smallsetminus T$. Then w is good, and thus J satisfies (1–3) by Observation 1.

Otherwise, let $T=xyz$ be the unique abnormal triple, and let $w \in X \smallsetminus T$. As T has no abnormal pairs, and at most one bad, normal pair, the remaining two pairs, say $xy, yz \subset T$, are good and normal. By Observation 1, some $J \in P := \{xy, yz\}$ satisfies (1–3), unless $C_0 \subseteq L(J)\cup L(w)$ for both $J \in P$. Then

$$C_0 \subseteq (L(xy)\cup L(w)) \cap (L(yz)\cup L(w)) = L(T)\cup L(w).$$

As T is abnormal, this yields the contradiction

$$l + u_2 \le |C_0| \le |L(T) \cup L(w)| = u_2 - 1 + l. \qquad \square$$

The next Lemma is needed for Step 11(b).

Lemma 14. *Suppose $b = 1$ and $X = xyzw \in \mathcal{U}_4$. If*

$$|W(xyz)| \le 2k + u_4 - 1 < |W(X)| \tag{3.30}$$

then $u_1 = 0$ and there exists a pair $J \subseteq xyz$ such that:

(1) $L(J) \nsubseteq C_3$, and so f_3 can be extended to an SDR of $\mathcal{L}(M_3 + v_J)$;
(2) $|W(\{v_J, v\})| \ge 2k$ for $v \in xyz \smallsetminus J$ and $|W(\{v_J, w\})| \ge 2k - 1 + u_3$; and
(3) $|W(\overline{J} + v_J)| \ge 2k + u_4$.

Proof. Consider a pair $vv' \subseteq xyz$. By (3.30) and (3.5),

$$\begin{aligned} 2k + u_4 - 1 &\ge |W(xyz)| \ge |W(vv')| \ge l(v) + l(v') - l(vv') \\ &\ge 2l - (l - k + u_2 + u_3) \ge 3k - 1 - k + u_1 + u_4 \\ &\ge 2k + u_1 + u_4 - 1. \end{aligned}$$

So $u_1 = 0$, $l(vv') = l - k + u_2 + u_3$, and $W(xyz) = W(vv')$. Since vv' is arbitrary, every color in $W(xyz)$ appears in at least two of the lists $L(x)$, $L(y)$, $L(z)$. So $|W(\{v_J, v\})| = |W(xyz)| \ge 2k$ for every pair $J \subseteq xyz$ and vertex $v \in xyz \smallsetminus J$. As $|C_3| < 2k \le |W(xyz)|$, there is a pair $J \subseteq xyz$ with $L(J) \nsubseteq C_3$. Furthermore, by (3.2),

$$|W(\{v_J, w\})| \ge l(J) + l(w) - l(J + w) \ge l - k + u_2 + u_3 + l - u_2 = 2k - 1 + u_3.$$

Finally, as $W(\{v_J, v\}) = W(xyz)$ for $v \in xyz \smallsetminus J$, and using (3.30),

$$|W(\overline{J} + v_J)| = |W(\{v_J, v\}) \cup W(w)| = |W(xyzw)| \ge 2k + u_4. \qquad \square$$

The next lemma completes the proof of our main theorem. The reader should keep Figure 3.1 and Table 1 in mind. For a part $X \in \mathcal{U}_4$ let X^* be the corresponding part of H. Then $X = X^*$ unless $b = 1 = \dot{u} + \ddot{u}$.

Lemma 15. *G' is L-choosable.*

Proof. First observe that if k is even then $b = \dot{u} = \ddot{u} = \dot{r} = 0$ and $H = G'$. In this case the following argument is much simpler.

We will prove $\mathcal{L}(V(H))$ has an SDR. Recall that $W := W(S) = \bigcup_{v \in S} L(v)$. Using Hall's Theorem it suffices to show $|S| \le |W|$ for every $S \subseteq V(H)$. Suppose for a contradiction that $|S| > |W|$ for some $S \subseteq V(H)$. We consider several cases, each of which assumes the previous cases fail.

Case 1: There is $X \in \mathcal{U}_4$ with $|S \cap X^*| = 4$. Since $|W| < |S| \le |H| \le |G'|$, Lemma 13 yields $b = 1$ and $|G'| - 1 = |W(X)| < |S| \le |H|$. Furthermore, Step 11(a) is executed, and so $|H| \le |G'| - 1$, a contradiction.

Case 2: There exists $X = xyzw \in \mathcal{U}_3$ with $|S \cap X'| = 3$. Since Case 1 fails,

$$|S| \le 3k - u_1 - u_2 - r_1 - r_2 - \dot{r}. \tag{3.31}$$

Say $I_X = xy$. By Step 5, $\Delta_1(xy) \ge 0$ and $l(xyz) + l(xyw) = 2u_2 - \Delta_2(xy)$. By (3.4), $l(xyz) + l(xyw) \le u_2 + r_1$ if $r_1 < l - k$; and it also holds if $r_1 = l - k$ since $u_2 \le 2k - l - u_3 \le l - k$ and $l(xyz) + l(xyw) \le 2u_2$. Thus

$$\begin{aligned} |W| &\ge |W(X')| \ge l(xy) + l(z) + l(w) - l(xyz) - l(xyw) - l(zw) \\ &= 2l + \Delta_1(xy) - 2u_2 + \Delta_2(xy) = 3k - b + \Delta_1(xy) - 2u_2 + \Delta_2(xy) \\ &\ge 3k - b + \Delta_1(xy) - u_2 - r_1 \ge |S| - b \ge |W|. \end{aligned} \tag{3.32}$$

So equality holds throughout. Thus (3.31) is sharp, $b=1$, $u_1=r_2=\dot r=\Delta_1(xy)=0$, and $Y'\subset S$ for all $Y\in\mathcal{R}_3$. Moreover, $r_1\le u_2$ since $\Delta_2(xy)\ge 0$, and $|W|=3k-1-u_2-r_1$.

Suppose $r_3\ge 1$. As $\dot Y'\subset S$, Lemma 10 implies

$$3k-b+\Delta_1^1(I_{\dot Y})-u_2-r_1\le |W(\dot Y')|\le |W|\le 3k-1-u_2-r_1.$$

Thus $\Delta_1^1(I_{\dot Y})=0$, and Step 7(a) is not executed. So Step 7(b) is executed. As $u_1=0=r_2$, Step 7(b*) is executed and $\dot r=1$. This contradiction shows $r_3=0$ and $r_1=l-k$.

As $X\in\mathcal{U}_3$,

$$l-k=r_1\le u_2\le 2k-l-u_3-u_4\le l-k-u_4.$$

So $u_4=0$, and by Step 5,

$$k=l-k+u_2+u_3\le l(xy)=l(\overline{xy})+\Delta_1(xy)=l(\overline{xy}).$$

By (3.1) this contradicts $u_1=0$.

Case 3: There exists $X=wxyz\in\mathcal{R}_3$ with $|S\cap X'|=3$. As the previous cases fail, $|S|\le 3k-u_1-u_2-u_3-r_1-r_2-\dot r$. By Step 7, $\Delta_1^1(I_X)\ge 0$, and by Lemma 10,

$$|W|\ \ge\ |W(X')|\ge 3k-b+\Delta_1^1(I_X)-u_2-r_1\ge |S|-b\ge |W|.$$

Thus $b=1$, $0=r_2=u_1=\dot r=\Delta_1^1(I_X)$, $|S|=3k-1-u_2-r_1$, and $\dot Y'\subseteq S$. Replacing X' with $\dot Y'$ yields $\Delta_1^1(I_{\dot Y})=0$. So Step 7(a) is not executed. As $0=r_2=u_1$, Step 7(b*) is executed. Thus $\dot r=1$, a contradiction.

Case 4: There exists $X\in\mathcal{U}_4$ with $|S\cap X^*|=3$. As the previous cases fail,

$$|S|\le 2k+u_4-\dot u-\ddot u$$

Let $xy\subseteq S\cap X$. By (3.5),

$$\begin{aligned}|W|\ &\ge\ l(x)+l(y)-l(xy)\ge 3k-b-(l-k+u_2+u_3)\\ &=\ 2k+(2k-l)-(u_2+u_3)-b\ge 2k+u_1+u_4-b\ge |S|-b\ge |W|.\end{aligned}$$

So $b=1$, $0=u_1=\dot u=\ddot u$, $|W|=2k+u_4-1$, and $|S|=2k+u_4$. Thus S has exactly three vertices in every class X^* with $X\in\mathcal{U}_4$, and exactly two vertices in every other class of H. In particular, $\dot Z'\subseteq S$. Since $\dot u=\ddot u=0$, we have $X=X^*$. As Step 11(a) is not executed, $|W(X)|\ge |G'|\ge |S|=2k+u_4$. Thus, as Step 11(b) is not executed, we have the contradiction

$$|W|\ge |W((S\cap X)\cup \dot Z')|\ge 2k+u_4=|S|.$$

Case 5: There exists $X\in\mathcal{U}_1$ with $|S\cap X'|=2$. Then $v_{I_X},v_{\overline{I}_X}\in S$. As the previous cases fail, $|S|\le 2k$. Now

$$|W|\ge L(v_{I_X})+L(v_{\overline{I}_X})\ge 2k\ge |S|.$$

Case 6: There is $X\in\mathcal{U}_3$ with $|S\cap X'|=2$. Say $S\cap X'=vv'$. As the previous cases fail, $|S|\le 2k-u_1$. If $v,v'\notin M$ then $\overline{I}_X=vv'$. By Step 5 and (3.1), $l(\overline{I}_X)\le k-1$. Thus

$$|W(vv')|\ge l(v)+l(v')-l(vv')\ge 2l-(k-1)\ge 2k\ge |S|.$$

Otherwise (say) $v=v_{xy}$; so $I_X=xy$ and $v'\notin M$. By Step 5, $l(v_{xy})\ge l-k+u_2+u_3$, and $l(xyv')\le u_2$ by (3.2). So

$$\begin{aligned}|W(vv')|\ &\ge\ l(v_{xy})+l(v')-l(xyv')\\ &\ge\ l-k+u_2+u_3+l-u_2\ge 2k-b+u_3\ge 2k\ge |S|.\end{aligned}$$

Case 7: There exists $X \in \mathcal{U}_4$ with $|S \cap X^*| = 2$. Say $S \cap X^* = vv'$. If possible, choose X so that $S \cap X^* \cap M = \emptyset$. As the previous cases fail, $|S| \leq 2k - u_1 - u_3$. If $v, v' \notin M$ then, using (3.5),

$$\begin{aligned} |W(vv')| &= l(v) + l(v') - l(vv') \geq 2l - (l - k + u_2 + u_3) \\ &= 2k - b + u_1 + u_4 \geq 2k \geq |S|. \end{aligned} \tag{3.33}$$

Else $b = 1$, and (say) $v \in M$. By Step 11, either $v = v_{I_{\dot{X}}}$ or $v = v_{I_{\ddot{X}}}$. By Lemmas 13 and 14, $u_1 = 0$.

If $v = v_{I_{\dot{X}}}$ then Step 11(a) was executed. So (i) $r_3 = 0$, (ii) $|W(vv')| \geq 2k - 1$, and (iii) if $|W(vv')| = 2k - 1$ then $|W(vv' \cup \dot{Z}') \cup C_3| \geq 2k$. Since

$$2k \geq |S| > |W| \geq |W(vv')| \geq 2k - 1,$$

we have $|S| = 2k$ and $u_3 = 0$. Thus S contains exactly two vertices of each part of H. As at most one class of $\mathcal{U}_4$ has merged vertices, the choice of X implies $u_4 = 1$; thus $u_2 = l - k$. Also, $\dot{Z}' \subseteq S$. Since $u_3 = 0 = r_3$, $M_3 \subseteq S$. So $|W| \geq |W(vv' \cup \dot{Z}') \cup C_3| \geq 2k$, a contradiction.

Otherwise $v = v_{I_{\ddot{X}}}$. Then Step 11(b) was executed. As only one part in $\mathcal{U}_4$ can have contracted vertices, $X = \ddot{X} = xyzw \in \mathcal{U}_4$ with (say) $I_{\ddot{X}} = xy$,

$$|W(xyz \cup \dot{Z}')| \leq 2k + u_4 - 1 < |W(\ddot{X})|,$$

$|W(\{v_{xy}, z\})| \geq 2k$ and $|W(\{v_{xy}, w\})| \geq 2k - 1 + u_3$. So we have the contradiction $|W(vv')| \geq 2k$ unless $v' = w$ and

$$2k \geq |S| > |W(\{v_{xy}, w\})| \geq 2k - 1 + u_3.$$

Thus $u_3 = 0$ and $|S| = 2k$. Again S contains exactly two vertices of each part of H, and the choice of X implies $u_4 = 1$. So $u_2 = l - k$. Also, $\dot{Z}' \subseteq S$. As $|W(\ddot{X})| > |W(xyz \cup \dot{Z}')|$, we have $|L(w) \smallsetminus W(xyz \cup \dot{Z}')| \geq 1$. So we have the contradiction

$$|W| \geq |W(\{v_{xy}, w\} \cup \dot{Z}')| \geq W(\dot{Z}') + 1 = l + u_2 + 1 = 2l - k + 1 = 2k.$$

Case 8: There exists $X = xyzw \in \mathcal{U}_2$ with $|S \cap X'| = 2$. Say $S \cap X' = \{v_I, w\}$. As the previous cases fail, $|S| \leq 2k - u_1 - u_3 - u_4 = l + u_2$. By Lemma 9, $L(xyz) \cap L(w) = \emptyset$, so

$$|W| \geq |W(X')| \geq l(xyz) + l(w) \geq u_2 + l \geq |S|.$$

Case 9: Otherwise. As the previous cases fail,

$$|S| \leq u_1 + u_2 + u_3 + u_4 + 2|\mathcal{R}| = l.$$

As $\mathcal{L}(M)$ has an SDR, there is a vertex $x \in S \smallsetminus M$. Thus $|W| \geq l(x) = l \geq |S|$. □

This completes the proof of Theorem 7. □

4. On-line Choosability

By Theorem 7, $\mathrm{ch}(K_{4*3}) = 4$. Using a computer we have checked that $\mathrm{ch}^{\mathrm{ol}}(K_{4*3}) = 5$, but do not have a readable argument to verify the upper bound. Here we prove the lower bound.

Theorem 16. $\mathrm{ch}^{\mathrm{ol}}(K_{4*3}) \geq 5$.

Proof. Figure 4.1 describes a strategy for Alice. The top left matrix depicts the initial game position, and Alice's first move. The positions in the matrix correspond to the vertices of K_{4*3} arranged so that vertices in the same part correspond to positions in the same column. The order of vertices within a column is irrelevant, as is the order of the columns. The numbers represent the size of the list of each corresponding vertex. The

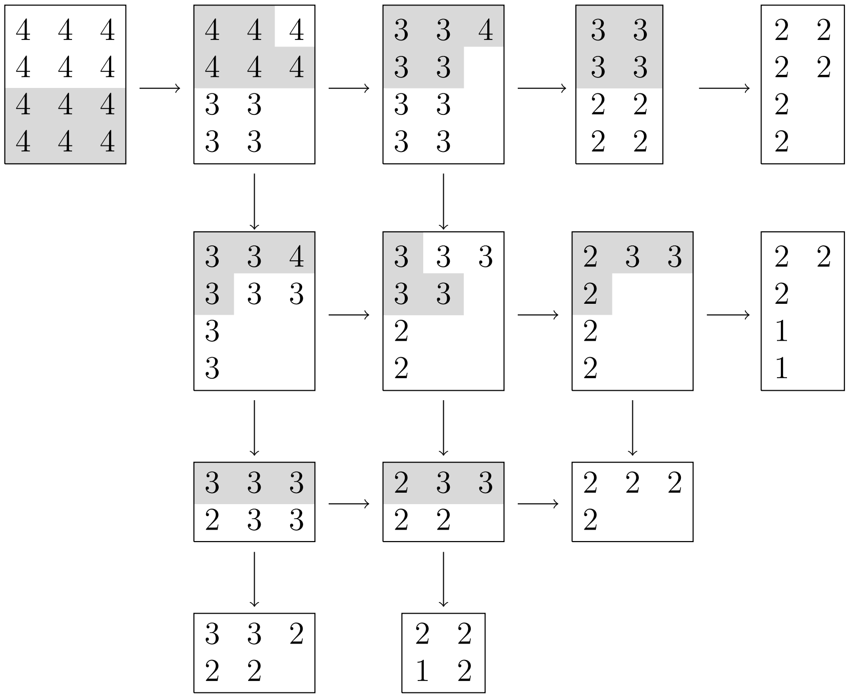

FIGURE 4.1. Strategy for Alice demonstrating $\mathrm{ch}^{\mathrm{ol}}(K_{4*3}) \geq 5$.

sequence of numbers represents a function f. The shaded positions represent the vertices that Alice presents on her first move.

As play progresses Bob chooses certain vertices presented by Alice and passes over others. When a vertex is chosen its position is removed from the next matrix (and the positions in its column of the remaining vertices and the order of the columns may be rearranged). When he passes over a vertex its list size is decreased by one (and its position in its column and the order of the columns may change). The arrows between the matrices point to the possible new game positions that arise from Bob's choice, not counting equivalent positions and omitting clearly inferior positions for Bob. In particular we assume Bob always chooses a maximal independent set.

For example, after Bob's first move there is only one possible game position, provided Bob chooses a maximal independent set. It is shown in the second column of the first row, along with Alice's second move. Now Bob has two possible responses that are pointed to by two arrows. Also consider the matrix in the third row and third column. There are three nonequivalent responses for Bob, but choosing the offered vertex in the second column of the matrix results in a position that is inferior to choosing the offered vertex in the first column. So this option is not shown.

Eventually, Alice forces one of five positions (G, f) such that G is not f-choosable, and Bob, being a gentleman, resigns. □

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