NOTE

Directed Triangles in Digraphs

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Received October 7, 1997

Let $c$ be the smallest possible value such that every digraph on $n$ vertices with minimum outdegree at least $cn$ contains a directed triangle. It was conjectured by Caccetta and Häggkvist in 1978 that $c = 1/3$. Recently Bondy showed that $c \leq (2\sqrt{6} - 3)/5 = 0.3797\ldots$ by using some counting arguments. In this note, we prove that $c \leq 3 - \sqrt{7} = 0.3542\ldots$

Let $G = (V, E)$ denote a digraph on $n$ vertices. The digraphs we consider here contain no loops or multiple arcs. We also assume that $G$ contains no digons; that is, if $(u, v) \in E$, then $(v, u) \notin E$. In 1978, Caccetta and Häggkvist [3] proposed the following conjecture:

**Conjecture 1.** Any digraph on $n$ vertices with minimum outdegree at least $r$ contains a directed cycle of length at most $\lceil nr \rceil$.

A particularly interesting special case that is still open is: any digraph on $n$ vertices with minimum outdegree at least $n/3$ contains a directed triangle. Short of proving this, one may seek a value $c$ as small as possible such that every digraph on $n$ vertices with minimum outdegree at least $cn$ contains a directed triangle. This was the strategy of Caccetta and Häggkvist [3], who showed that $c \leq (3 - \sqrt{5})/2 = 0.3819\ldots$ by a simple inductive argument. Recently Bondy [2] showed that $c \leq (2\sqrt{6} - 3)/5 = 0.3797\ldots$ by using some counting arguments. In Theorem 1, we use induction and combine techniques from [2, 4] to show that $c \leq 3 - \sqrt{7} = 0.3542\ldots$

**Theorem 1.** If $n = 3 - \sqrt{7} = 0.3542\ldots$, then any digraph on $n$ vertices with minimum outdegree at least $3n$ contains a directed triangle.

**Proof.** We use induction on $n$. Clearly Theorem 1 holds for $n = 3$. Now assume that Theorem 1 holds for all digraphs with fewer than $n$ vertices and $G$ is a counterexample with $n$ vertices. Without loss of generality,
it may be supposed that $\deg^+(u) = r = \lceil \alpha n \rceil$ for all $u \in V$. Let $N^+(u) = \{ v \in V : (u, v) \in E \}$ and $N^-(u) = \{ v \in V : (v, u) \in E \}$.

For any arc $(u, v) \in E$, we set:

\[ p(u, v) := [N^+(v) \setminus N^+(u)], \]
\[ q(u, v) := [N^-(u) \setminus N^-(v)], \]
\[ t(u, v) := [N^+(u) \cap N^+(v)], \]

the number of induced directed 2-paths whose first arc is $(u, v)$; the number of induced directed 2-paths whose last arc is $(u, v)$; the number of transitive triangles having the arc $(u, v)$ as “base.”

We claim that

\[ n > r + \deg^-(v) + q(u, v) + (1 - \alpha) t(u, v). \quad (1) \]

If $t(u, v) = 0$, then (1) holds because $N^+(v)$, $N^-(v)$, and $N^-(u) \setminus N^-(v)$ are pairwise-disjoint sets of cardinalities $r$, $\deg^-(v)$, and $q(u, v)$, respectively. If $t(u, v) > 0$, some vertex $w \in N^+(u) \cap N^+(v)$ has outdegree less than $\alpha t(u, v)$ in the subdigraph of $G$ induced by $N^+(u) \cap N^+(v)$ (otherwise this subdigraph would contain a directed triangle, by the minimality of $G$). Thus $w$ is joined to at least

\[ \deg^+(w) - p(u, v) - \alpha t(u, v) = \deg^+(v) - p(u, v) - \alpha t(u, v) = (1 - \alpha) t(u, v) \]

vertices not in $N^+(v)$. Since $G$ has no directed triangle, these vertices are neither in $N^-(v)$ nor in $N^-(u) \setminus N^-(v)$. This establishes the claim. Noting that $t(u, v) = r - p(u, v)$, we can rewrite inequality (1) as

\[ \alpha t(u, v) > 2r - n + \deg^-(v) + q(u, v) - p(u, v). \]

We sum this inequality over all $(u, v) \in E$.

\[ \sum_{(u, v) \in E} \alpha t(u, v) = \alpha t, \]

where $t$ is the number of transitive triangles in $G$,

\[ \sum_{(u, v) \in E} (2r - n) = rn(2r - n), \]

\[ \sum_{(u, v) \in E} \deg^-(v) = \sum_{v \in V} (\deg^-(v))^2 \geq \frac{1}{n} \left( \sum_{v \in V} \deg^-(v) \right)^2 = r^2 n \]

and

\[ \sum_{(u, v) \in E} (q(u, v) - p(u, v)) = 0, \]
because \( \sum_{(u,v) \in E} q(u,v) \) and \( \sum_{(u,v) \in E} p(u,v) \) are both equal to the number of induced directed 2-paths in \( G \). Thus \( \alpha t > m(3r-n) \). But \( t \leq h(n/\alpha^2) \), the number of out-2-claws of \( G \), so \( \alpha > 6 - 2n/r \geq 6 - 2/\alpha \), that is, \( \alpha < 3 - \sqrt{7} \).

Graaf, Schrijver, and Seymour [5] considered a similar problem that involved both the minimum outdegree and the minimum indegree. They proved that if \( \beta = 0.3487... \), then any digraph on \( n \) vertices with both minimum outdegree and minimum indegree at least \( \beta n \) contains a directed triangle. In fact, they showed that once a value of \( \alpha \) is found, then a value of \( \beta \) can be obtained from the inequality

\[
\left( \frac{4}{\alpha^2} - \frac{2}{\alpha} \right) x^2 - \left( \frac{24}{\alpha^3} - \frac{16}{\alpha} \right) x + \left( \frac{36}{\alpha^3} - \frac{30}{\alpha} + 1 \right) > 0 \quad [5, \text{pp. 282, formula (16)}].
\]

Therefore by using \( \alpha = 3 - \sqrt{7} \), one can obtain the following slight improvement.

**Corollary 1.** If \( \beta = 0.3477... \), then any digraph on \( n \) vertices with both minimum outdegree and minimum indegree at least \( \beta n \) contains a directed triangle.

**ACKNOWLEDGMENT**

I thank Professor J. A. Bondy for kindly providing me with a simpler version of my original proof for Theorem 1.

**REFERENCES**