Graph Theory Final Exam

May 12, 2004

Directions. Solve the five problems below. Ask questions whenever it is not clear what is being asked of you. Each problem is worth 20 points.

Notation. For a positive integer n let $[n] = \{1, 2, \ldots, n\}$. For $0 \le k \le n$ define $\binom{[n]}{k}$ to be the family of subsets of $[n]$ having size k. For a graph G let $\Delta(G)$ denote its maximum degree.

1. Let $r \le n$ be positive integers. Prove that if $G = (V, E)$ is an n-vertex graph containing no clique on $r + 1$ vertices and having as many edges as possible, then G is r-partite.

2. Let $G = (V, E)$ be a graph with $A, B \subseteq V$. Suppose that the size of a minimum A, B-separating set is k. Prove that there exist k disjoint A, B-paths.

3. Let G be a 3-connected graph and let L be a list assignment for G such that all lists have size $\Delta = \Delta(G)$. Prove that if G does not contain a $(\Delta + 1)$-clique then G is L-colorable. You may use the fact that for any connected graph H and any vertex w of H, the vertices of H can be ordered as $w_1, \ldots, w_n = w$ so that for every index $i \in [n-1]$ there exists an index $j \in [n]$ such that $i < j$ and $v_i \sim_H v_j$. **Caution:** Two lists might not have a common color.

4. Let $1 \le d_1 \le d_2 \le \cdots \le d_n$ be integers, with $2 \le n$. Prove that there exists a tree $G = (\{v_i : i \in [n]\}, E)$ having degrees $d(v_i) = d_i$ iff $\sum_{i \in [n]} d_i = 2n - 2$.

5. Let $n = 2k + 1$, where k is a positive integer. Prove that there exists a one-to-one function $f : \binom{[n]}{k} \to \binom{[n]}{k+1}$ such that $S \subseteq f(S)$ for every $S \in \binom{[n]}{k}$.

Graph Theory Qualifier Exam

August 12, 2005

Directions. Solve the five problems below. Ask questions if any of the wording is ambiguous or confusing. Each problem is worth 20 points.

1. Let $G = (V, E)$ be a graph and let $A, B \subseteq V$. Show that if k is the size of the smallest set of vertices that separates A from B then there exist k disjoint paths from A to B.

2. Prove that a connected, plane multigraph with n vertices, m edges and f faces satisfies

$$n - m + f = 2 \ .$$

3. Suppose that G is a bipartite graph with bipartition $\{X, Y\}$ that does not contain the cycle C_4 on four vertices. Show that if every vertex in X has degree at least $\frac{3}{2}a$ and $|X| \le a^2$ then G has a matching that saturates X.

4. Let T be a tree with an even number of vertices. Prove that T has a *unique* spanning subgraph in which every vertex has odd degree.

5. Let $G = (V, E)$ be a graph with vertices $v_1, \ldots, v_n$. For each $1 \le j \le n$ define

$$B(j) = \{i \mid i < j, v_i \sim v_j\} \ .$$

For $1 \le i < j < k \le n$ we say the triple (i, j, k) is *dangerous* if $v_i \sim v_k \sim v_j$ (v_k adjacent to both v_i and v_j). For each $1 \le j \le n$ define

$$D(j) = \{i \mid i < j, (i, j, k) \text{ dangerous for some } k > j\} \ .$$

Suppose that $|B(j) \cup D(j)| < t$ for all $1 \le j \le n$.

(a) Let C be a cycle of G. Prove that C contains a dangerous triple T.

(b) Prove that G can be properly t-colored so that no cycle is 2-colored.

Graph Theory Qualifier Exam

May 04, 2006

Directions. Solve the five problems below. Ask questions if any of the wording is ambiguous or confusing. Each problem is worth 20 points.

1. Let $G = (V, E)$ be simple graph on n vertices. Show that if G does not contain an $(r+1)$-clique then

$$|E| \leq |E(T_{n,r})|$$

where $T_{n,r}$ is Turan graph (i.e. an r-partite graph on n vertices such that for any two partite sets A, B, $||A| - |B|| \leq 1$).

2. Prove that a connected, plane multigraph with n vertices, m edges and f faces satisfies

$$n - m + f = 2 \ .$$

3. Suppose that G is a simple bipartite graph with bipartition $\{X, Y\}$ that does not contain the cycle C_4 on four vertices. Show that if every vertex in X has degree at least $\frac{3}{2}a$ and $|X| \leq a^2$ then G has a matching that saturates X.

4. Let T be a tree with an even number of vertices. Prove that T has a spanning subgraph in which every vertex has odd degree.

5. Let $G = (V, E)$ be a simple graph with vertices $v_1, \ldots, v_n$. For each $1 \leq j \leq n$ define

$$B(j) = \{i \mid i < j, v_i v_j \in E\} \ .$$

For $1 \leq i < j < k \leq n$ we say the triple (i, j, k) is *dangerous* if $v_i v_k \in E$ and $v_k v_j \in E$ (v_k adjacent to both v_i and v_j). For each $1 \leq j \leq n$ define

$$D(j) = \{i \mid i < j, (i, j, k) \text{ dangerous for some } k > j\} \ .$$

Suppose that $|B(j) \cup D(j)| < t$ for all $1 \leq j \leq n$.

(a) Let C be a cycle of G. Prove that C contains a dangerous triple T.

(b) Prove that G can be properly colored using t-colors so that no cycle is 2-colored.

6. Let G be a graph with an even number of vertices such that G is r-connected and does not contain $K_{1,r+1}$ as an induced subgraph. Show that G has a perfect matching.
7. State and prove König-Egervary Theorem.

Graph Theory Qualifier

May 2007

1. (20) (Dirac's Theorem) Prove that if G is a graph with $\delta(G) \geq \frac{1}{2}|G|$ then G has a hamiltonian cycle.

2. (20) (Vizing's Theorem) Prove that every graph G satisfies $\chi'(G) \leq \Delta(G)+1$.

3. (20) Suppose $k \geq 2$ and G is a k-connected graph without $K_{1,3}$ as an induced subgraph. Show that G contains a cycle with length at least $\min\{|G|, 4k\}$.

4. (20) Let G be a balanced X,Y-bigraph. Suppose that all $x \in X$, $y \in Y$ with $xy \notin E(G)$ satisfy $d(x) + d(y) \geq |X|$. Show that G has a 1-factor.

5. Let G be a bipartite, planar graph.

   (a) (5) Use Euler's formula to prove that $\|G\| \leq 2|G| - 4$, if $|G| \geq 3$.

   (b) (3) Prove that G has a vertex v with $d(v) \leq 3$.

   (c) (6) Prove that if $\vec{G}$ is an orientation of G and $d^+_{\vec{G}}(v) = 3$ then there is a directed path from v to a vertex w with $d^+_{\vec{G}}(w) \leq 1$.

   (d) (6) Using the fact that every orientation of every bipartite graph has a kernel, prove that $\chi'_l(G) \leq 3$.

Graph Theory Qualifier Exam

August 10, 2007

1. Prove that every bridgeless cubic graph has a 1-factor.

2. (Menger's Theorem) Let G be a graph. Prove that for all $A, B \subseteq V(G)$ the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B. (If you prefer, you may state and prove an equivalent form of the Theorem.)

3. Let G be a graph such that $d(x) + d(y) \leq 2k - 1$ for every edge xy. Prove that G is k-choosable and show that this is tight, i.e., for all $k \geq 1$ find a graph H such that $d(x) + d(y) \leq 2k$ for every edge xy and G is not k-choosable.

4. Let G be a connected even graph. Show that there is an orientation $\vec{G}$ of G such that for every vertex r there is a spanning tree $\vec{T}$ of $\vec{G}$ so that, for every vertex v, the unique vr-path in $\vec{T}$ is oriented from v to r. [Hint: For the right orientation, consider a maximum subtree with an appropriate property.]

5. A graph G is *outerplanar* if it can be drawn in the plane so that every vertex is on the outer face. Show that G is outerplanar iff it contains neither a subdivision of K_4 nor a subdivision of $K_{2,3}$. [Hint: Consider adding a new vertex to G.]

6. Let G be an A, B-bigraph satisfying $\delta(G) \geq k$ and $|A| = 4k = |B|$. Suppose further that there exist $X \subseteq A$ and $Y \subseteq B$ such that $|X| \geq 2k$, $|Y| \geq 2k$, and every vertex in $X \cup Y$ has degree at least $3k$. Show that G has a 1-factor.

Graph Theory Qualifier

May 1, 2008

1. Prove that for any graph G with $\omega(G) \leq r$ there exists an r-partite graph H satisfying $\|G\| \leq \|H\|$.

2. Prove that if a graph is 3-connected and contains neither a subdivision of $K_{3,3}$ nor a subdivision of K_5 then it is planar.

3. Let G be a 3-connected graph with $\omega(G) \leq \Delta := \Delta(G)$. Let L be a list assignment for G such that all lists have size Δ. Prove that G is L-colorable. You may use the fact that for any connected graph H and any vertex w of H, the vertices of H can be ordered as $w_1, \ldots, w_n = w$ so that every vertex, except w, has a neighbor with higher index. **Caution:** Two lists might be disjoint.

4. Let G be an X,Y-bigraph. Suppose every edge xy with $x \in X$ satisfies $d(x) \geq d(y)$. Prove that G has a matching saturating X. [Hint: Consider the weight function $w : E(G) \to \mathbb{Q}$ defined by $w(xy) := \frac{1}{d(x)}$. Count $\sum_{e \in E(H)} w(e)$ in two ways for appropriate subgraphs H.]

5. Let G be a planar graph with $\delta(G) = 5$.

   (a) Let $S \subseteq V(G)$. Prove that if H is a component of $G - S$ with $|H| = 3$ then $|N_G(V(H)) \cap S| \geq 4$.

   (b) Use (a) to prove that G has a matching with at most $\frac{1}{5}|G|$ unsaturated vertices. You may use standard facts about the number of edges in planar graphs.

Graph Theory Qualifying Exam
09/02/2008

Solve all of the five problems. Assume that all graphs are simple.

1. (Thomassen's Theorem) Prove that every planar graph is 5-choosable.
2. (Hall's Theorem) Let G be an X, Y-bigraph. Prove that G has a matching that saturates X if all $S \subseteq X$ satisfy $|S| \leq |N(S)|$.
3. For every positive integer k construct a triangle free graph G with $\chi(G) \geq k$. (For partial credit you may do the case $k = 4$.)
4. Prove that if G is a k-regular, $(k-1)$-edge-connected graph on an even number of vertices then G has a 1-factor. (For partial credit, you may assume that G is k-edge-connected.)
5. Prove that a k-connected graph with no hamiltonian cycle has an independent set of size $k + 1$.

Graph Theory Qualifying Exam
05/12/2009

Solve all five problems. Assume that graphs are simple.

1. (Menger's Theorem) Let G be a graph. Prove that for all $A, B \subseteq V(G)$ the minimum number of vertices separating A from B is equal to the maximum number of disjoint paths from A to B. (If you prefer, you may state and prove an equivalent form of the Theorem.)

2. (König-Egerváry) Show that in a bipartite graph the size of the maximum matching is equal to the size of the minimum cover.

3. Show that a k-connected graph of even order which has no $K_{1,k+1}$ as an induced subgraph has a perfect matching.

4. A graph G is called outerplanar if it is possible to draw G in the plane so that every vertex lies on the boundary of the outer face. Show that an outerplanar graph is 3-choosable.

5. Let G be a connected graph on $n \geq 4$ vertices with m edges. Show that if $m \geq 2n - 2$ then G has two cycles of equal length. (*Hint: Use a spanning tree of* G *to count cycles.*)

Graph Theory Qualifier

May 2010

1. Prove that for every positive integer $k \geq 3$ there exists a graph G on $\lfloor 2^{\frac{k-1}{2}} \rfloor$ vertices such that both $\alpha(G) < k$ and $\omega(G) < k$.

2. Prove Menger's Theorem: Let $G = (V, E)$ be a graph and $A, B \subseteq V$. The minimum size of an A, B-separating set is equal to the maximum number of disjoint A, B-paths.

3. Suppose $G = K_{2t}$ and M is a 1-factor of G. Show that if L is a t-list assignment for G such that $L(u) \cap L(v) = \emptyset$ for all $uv \in M$ then G is L-colorable. [Hint: System of distinct representatives.]

4. Let $G = (V, E)$ be a connected graph with a special vertex $r \in V$. For a spanning tree T of G and a vertex $x \in V$ let rTx be the unique r, x-path in T. Call T *normal* if for all $xy \in E$ either $x \in V(rTy)$ or $y \in V(rTx)$. Prove that G has a normal spanning tree. [Hint: Arguing by induction, prove that if P is a path in G starting at r then G has a normal spanning tree that contains P. Consider a maximal path extending P.]

5. Prove that every 3-connected, non-planar graph on at least six vertices contains a subdivision of $K_{3,3}$.

Graph Theory Qualifier Exam

January 18, 2011

(1) Let $G := (V, E)$ be a graph, and $S \subseteq V$ be a subset. Set $\mathcal{C} := \mathcal{C}_{G-S}$, the set of components of $G-S$, and set $q := o(G-S)$, the number of odd components of $G-S$. Let $\mathcal{H}$ be the $S, \mathcal{C}$-bigraph such that $xC \in E(\mathcal{H})$ if $x \in S$, $C \in \mathcal{C}$ and $N_G(x) \cap V(C) \neq \emptyset$. Suppose that S has been selected so that:

(i) $q - |S|$ is maximum and

(ii) subject to (i), $|S|$ is maximum.

Prove:

(a) $|C|$ is odd for all components $C \in \mathcal{C}_{G-S}$;

(b) There is a matching $M \subseteq E(\mathcal{H})$ that saturates (matches) every vertex in S.

(2) Let $G = (V, E)$ be a graph with a minimum separating 2-set $S := \{x, y\}$ that contains neither a TK_5 nor a $TK_{3,3}$. Prove

(a) $G + xy$ contains neither a TK_5 nor a $TK_{3,3}$;

(b) use (a) to prove that G is planar, provided that for all graphs H, if H contains neither a TK_5 nor a $TK_{3,3}$ and $\|H\| < \|G\|$, then H is planar.

(3) Suppose $G := (V, E)$ is a graph on $n \geq 3$ vertices such that $d(x) + d(y) \geq n$ for all $xy \notin E$. Prove that G has a hamiltonian cycle.

(4) Let L be a k-list assignment for a 2-connected graph $G := (V, E)$ with $\Delta(G) \leq k$. Prove that if there are two vertices x and y such that $L(x) \neq L(y)$ then G is L-colorable.

(5) Show that every tree T has a vertex $v \in V(T)$ such that for every edge $e \in E(T)$ the component of $T - e$ that contains v has order at least $|T|/2$.

Graph Theory Qualifier

May 10, 2011

1. (15 pts.) Prove that for all graphs G and positive integers a and b if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.

2. (15 pts.) Recall that $T_{n,r}$ is the r-partite graph on n vertices such that every part has size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$. Prove that among all graphs $G = (V, E)$ on n vertices with $\omega(G) \leq r$, the one with the most edges is $T_{n,r}$. (You may use the fact that this is true for all r-partite graphs.)

3. (15 pts.) Let G be an X, Y-bigraph with $|X| = |Y| = k$ and $\delta(G) \geq \frac{k}{2}$. Prove that G has a 1-factor.

4. (25 pts.) Let G be a planar bipartite graph.

   a) Use Euler's Formula to prove that $\|G\| \leq 2|G| - 4$.

   b) Suppose G has an orientation with $d^+(r) \geq 3$ for some vertex $r \in V$. Let $W \subseteq V$ be the set of all vertices w (including r) such that there is a directed path from r to w. Prove that $H = G[W]$ has a vertex v with $d^+_H(v) \leq 1$.

   c) Show that G has an orientation with maximum out-degree at most 2. [Hint: Argue by induction and consider reversing a directed path.]

   d) It is known that every directed graph without a directed odd cycle has a kernel. Use this to show that G is 3-list colorable.

   e) Construct a small example (including lists) of a graph that is bipartite and planar, but not 2-list colorable.

5. (15 pts.) Let $G = (V, E)$ be a nonplanar 3-connected graph with at least six vertices. Show that G contains a subdivision of $K_{3,3}$.

6. (15 pts.) Let $G = (V, E)$ be a 2-edge-connected graph. Show that G has a spanning 2-edge-connected subgraph with $\|H\| \leq 2|G| - 2$. [Hint: Start with a spanning tree T and add a small set $F \subseteq E$ to T so that $T + F - e$ is connected for every $e \in E(T)$.]

Graph Theory Qualifier

August 16, 2011

1. Prove that every planar graph is 5-choosable.

2. Prove that every 3-connected graph G with $|G| \geq 5$ has an edge e such that $G \cdot e$ is 3-connected.

3. Let $d_1, \ldots, d_n$ be positive integers with $n \geq 2$ such that $\sum_{i=1}^{n} d_i = 2n - 2$. Prove that there exists a tree on n vertices with degree sequence $d_1, \ldots, d_n$.

4. Let G be a 4-connected planar graph with $|G|$ even. Prove that G has a perfect matching. You may use the fact that every bipartite planar graph H satisfies $\|H\| \leq 2|H| - 4$.

5. Let G be an X, Y-bigraph. Prove that every orientation D of G has a kernel. [Hint: When is X a kernel?]

Graph Theory Qualifier

May 1, 2012

1. Prove: Every simple planar graph is 5-list colorable.

2. Recall that $T_{n,r}$ is the complete r-partite graph on n vertices such that every part X satisfies $\lfloor \frac{n}{r} \rfloor \leq |X| \leq \lceil \frac{n}{r} \rceil$. Prove: Among all graphs $G = (V, E)$ on n vertices with $\omega(G) \leq r$, the one with the most edges is $T_{n,r}$.

3. Recall that a graph is transitive if for every pair of vertices x, y there is an automorphism of G that sends x to y. Prove: Every connected, transitive graph with an even number of vertices has a perfect matching. (Be sure you use all three hypotheses.)

4. Let $k, n \in \mathbb{N}$, and suppose G is an A, B-bigraph with $|A| = n = |B|$ such that $\delta(G) \geq k$ and for all $X \subseteq A, Y \subseteq B$, if $|X|, |Y| \geq k$ then $|E(X, Y)| \neq \emptyset$. Prove: G has a perfect matching.

5. Recall that an x, X-fan is a set of $|X|$ internally disjoint x, X-paths whose ends in X are distinct. Let $G = (V, E)$ is a graph with $x \in V$ and $Y, Z \subseteq V$ and $k = |Y| = |Z| - 1$. Suppose $\mathcal{Q} = \{Q_y : y \in Y\}$ is an x, Y-fan in G, where each Q_y is an x, y-path. Similarly, suppose $\mathcal{R} = \{R_z : z \in Z\}$ is an x, Z-fan in G, where each R_z is an x, z-path. Prove: There exists an $x, (Y + z)$-fan in G for some $z \in Z$. [Hint: Add a new vertex w whose neighborhood is Z and apply a theorem.]

Graph Theory Qualifier

August 21, 2012

Directions. Solve **all** six problems below. Ask questions if any of the wording is ambiguous or confusing.

1. (10 pts.) Prove that for every integer $k \geq 2$ there exists a graph G with $\alpha(G), \omega(G) < k$ and $|G| \geq 2^{(k-1)/2}$.

2. (10 pts.) Show that if G is a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$, then G contains a Hamilton cycle.

3. (10 pts.) Let $G = (V, E)$ be a planar graph with $\delta(G) = 5$.

   (a) Suppose $S \subseteq V$, and H be a component of $G - S$ with $|H| = 3$. Show that $|\bigcup_{x \in V(H)} (S \cap N(x))| \geq 4$.

   (b) Prove that G has a matching of size at least $\frac{2}{5}|G|$, i.e., at most $\frac{1}{5}|G|$ unsaturated vertices.

4. (10 pts.) Prove: Every tree T containing $2k$ vertices with odd degree (and maybe some vertices with even degree) decomposes into k paths.

5. (10 pts.) Suppose $G := K_{s,t}$ is an S, T-bigraph, where $|S| = s$. Prove: G is s-choosable iff $t < s^s$. [Hint: First prove that if L is an s-list assignment for G with $L(v) \cap L(w) \neq \emptyset$ for distinct $v, w \in S$ then G is L-colorable.]

6. (10 pts.) Suppose G is a three connected graph. Prove: For all vertices $x, y \in V(G)$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ is an x, y-path and $G[V_2]$ is connected. [Hint: Contract an edge.]