Graph Theory Notes

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CHAPTER 1

Introduction

This is an on going project. Before each lecture there will likely be changes to the material not yet covered, especially before each lecture. There may also be changes immediately after a lecture based on class questions, etc. Later corrections to previous material will be marked in red. For these reasons, you should continuously download the current version.

Use at your own risk. There are bound to be typos. Corrections are appreciated, and rewarded with small amounts of extra credit, especially when they indicate mathematical understanding. But you are also encouraged to ask questions in emails about things you do not understand. I will try to answer questions quickly, but if they are complicated, or there are many, I may need to wait until the next class.

These notes are meant to enhance, not replace, the lectures and class discussions. They are intended to be concise records of proofs, to free students from the need to take careful notes during class. However the motivation for these proofs is left to the lectures and discussions in class.

1.1. Graphs

Formally a *graph* is an ordered pair $G = (V, E)$ where E is an irreflexive, symmetric, binary relation on V. Since E is symmetric there is no need to keep track of the order of pairs $(x, y) \in E$; since it is irreflexive there are no ordered singletons (x, x) in E. This leads to a more intuitive formulation. We take E to be a set of unordered pairs of elements from V. Elements of V are called *vertices*; elements of E are called *edges*. If $x, y \in V$ are vertices and $\{x, y\} \in E$ is an edge we usually (but not always) denote $\{x, y\}$ by the shorthand notation xy. So $xy = \{x, y\} = \{y, x\} = yx$. The vertices x and y are called *ends*, or *endpoints*, of the edge xy. The ends x and y of an edge xy are said to be *adjacent* and the end x is said to be *incident* to the edge xy. We also say that x and y are *joined* or *linked* (not connected) by the edge xy. Two edges are said to be *adjacent* if they have a common end. In this course, all graphs have a finite number of vertices, unless it is explicitly stated that they have infinitely many. Graphs are illustrated by drawing dots for vertices and joining adjacent vertices by lines or curves.

Our definition of graph is what West calls a *simple graph*. Most of the time we will only be interested in simple graphs, and so we begin with the simplest definition. When necessary, we will introduce the more complicated notions of *directed graphs* and *multigraphs,* but here is a quick hint. A directed graph $G = (V, E)$ is any binary relation (not necessarily irreflexive or symmetric) on V. In other words E is any set of *ordered* pairs of vertices. A *multigraph* is obtained by letting by letting E be a multiset; then two vertices can have more than one edge between them. A *hypergraph* is obtained by letting E be a set of subsets of V, where the elements of E can have any size. If they all have size k then we get a k*-uniform hypergraph*, also called a k-graph. So ordinary graphs are 2-uniform hypergraphs.

The study of graph theory involves a huge number of of parameters—see the front and back inside covers of West or the last two pages of Diestel. This can be quite daunting. My strategy is to introduce these parameters as they are needed. Please feel free to interrupt lectures to be reminded of their meanings. Most of the time my notation will agree with West, and I will try to emphasize differences. Next we introduce some very basic notation.

Given a graph G, $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges of G. Set $|G| := |V(G)|$ and $\|G\| := |E(G)|$; this is not standard, and instead the book uses $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Suppose $v \in V(G)$ is a vertex of G. Define

$$N_G(v) = \{w \in V(G) : vw \in E(G)\}; \qquad E_G(v) = \{e \in E(G) : v \text{ is an end of } e\}.$$

The set $N_G(v)$ is called the (open) *neighborhood* of v, and its elements are called *neighbors* of v. So a vertex w is a neighbor of v iff it is adjacent to v. When there is no confusion with other graphs the subscript G is often dropped. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$—we dropped the subscript. The set $E_G(v)$ is the set of edges incident to v; again, we may drop the subscript G. The text does not provide notation for this set. For simple graphs $|N(v)| = |E(v)|$. However for multigraphs this may not hold, since two vertices might be joined by several edges. With this in mind, define the degree of a vertex v to be $d_G(v) := |E_G(v)|$, but note that for simple graphs $d_G(v) = |N_G(v)|$. A graph is *k-regular* if every vertex has degree k.

We use the following set theoretic notation. The sets of natural numbers, integers and positive integers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z}^+$. For integers a and b let $\{a, \dots, b\}$ indicate the set $\{i \in \mathbb{N} : a \leq i \leq b\}$. Then $\{0, \dots, -1\} = \emptyset$. For $n \in \mathbb{N}$ set $[n] := \{1, \dots, n\}$; in particular $[0] = \emptyset$. For a set X and an element y, set $X + y := X \cup \{y\}$ and $X - y := X \smallsetminus \{y\}$. Finally, $\binom{X}{n}$ is the set of all n-element subsets of X.

In this course all graphs G satisfy $1 \leq |G| < \infty$, unless a specific exception is stated.

1.2. Proofs by Mathematical Induction

Most proofs in graph theory involve mathematical induction, or at least the Least Element Axiom. Here we quickly review this technique. Also see the discussion in West on pages 19–20, and especially the *induction trap* on page 42.

Here we present mathematical induction in terms of the Least Element Axion (LEA). First we give a careful definition of "least element".

Definition 1. Let $B \subseteq \mathbb{N}$ be a set of natural numbers. A number $l \in \mathbb{N}$ is a least element of B if

(L1) $\{0, \dots, l-1\} \subseteq \mathbb{N} \smallsetminus B$, and
(L2) $l \in B$.

The following axiom (LEA) is fundamental.

Axiom (LEA). *Every nonempty set of natural numbers has a least element.*

Consider a set of "good" natural numbers S, and let $B = \mathbb{N} \smallsetminus S$ be the set of "bad" numbers. We would like to prove that every natural number is "good", i.e., $S = \mathbb{N}$. Here is a way to organize the argument.

Theorem 2 (Principle of Induction). *Suppose $S \subseteq \mathbb{N}$. Then $S = \mathbb{N}$, if*

$$\forall n \in \mathbb{N}\ (\{0, \dots, n-1\} \subseteq S \to n \in S). \tag{1.2.1}$$

Proof. Suppose $S \subseteq \mathbb{N}$ satisfies (1.2.1). As $S \subseteq \mathbb{N}$ we only need prove $\mathbb{N} \subseteq S$. Arguing by contradiction, assume $\mathbb{N} \nsubseteq S$. Then $B := \mathbb{N} \smallsetminus S \neq \emptyset$. By LEA, B has a least element l. By (L1) applied to l, $\{0, \dots, l-1\} \subseteq \mathbb{N} \smallsetminus B = S$. By (1.2.1), $l \in S$, so $l \notin \mathbb{N} \smallsetminus S = B$. This contradicts (L2). □

Using the Principle of Induction to prove that $S = \mathbb{N}$, we prove (1.2.1). For this we consider *any* natural number n. If $\{0, \dots, n-1\} \nsubseteq S$ then we are done, so we suppose $\{0, \dots, n-1\} \subseteq S$. We use this "induction hypothesis" to prove that $n \in S$. If $n = 0$ then there is no natural number $k < n$, so $\{0, \dots, n-1\} = \emptyset \subseteq S$; thus the induction hypothesis has given us no new information. Usually the case $n = 0$ is special; more generally, we say the cases of the argument that do not use the existence in S of a smaller number than n form the *base step*. **Thus the case $n = 0$ is always part of the base step, but the base step may include more cases.** The cases that use that some smaller number than n is in S form the *induction step*. Here is an example. Notice that the statement of the theorem is carefully phrased because 0 and 1 do not have prime factors. This must be reflected in the definition of the set S.

Proposition 3. *Every natural number greater than* 1 *has a prime factor.*

Proof. Let $S = \{n \in \mathbb{N} : n \leq 1 \text{ or } n \text{ has a prime factor}\}$. It suffices to show (1.2.1). Consider any $n \in \mathbb{N}$ such that $\{0, \dots, n-1\} \subseteq S$. We must show $n \in S$. If $n \leq 1$ then $n \in S$ by definition. So suppose $n \geq 2$. If n is prime then it is a prime factor of itself, and so it is in S. Otherwise, there exist integers a, b such that $1 < a, b < n$ and $ab = n$. Since $a < n$, we have $a \in S$. Since $1 < a$ this means that a has a prime factor p. Since p is a factor of a and a is a factor of n, p is a (prime) factor of n. □

In the above argument, the cases $n \leq 1$ and n prime form the base step. Notice that in the induction step we never used that $n-1$ was in S; we only used that the factor a was in S, and it is easily seen that a is always less than $n-1$.

Here is an example from graph theory. First we need some additional notation. For a graph G, let $\delta(G) = \min\{d(v) : v \in V(G)\}$. We say that a graph contains a cycle if it has a sequence of at least three distinct vertices $v_1, \dots, v_s$ such that $v_i v_{i+1} \in E$ for all $i \in [s-1]$ and $v_s v_1 \in E$; in this case we call $v_1 \dots v_s v_1$ a cycle. The following (typical) statement is about all graphs, not all natural numbers, so it may be surprising that we can prove it by induction.

Proposition 4. *Every graph $G = (V, E)$ with $\delta(G) \geq 2$ contains a cycle.*

Proof. Let $S = \{n \in \mathbb{N} : \text{all graphs } G \text{ with } |G| = n \text{ and } \delta(G) \geq 2 \text{ have a cycle}\}$. Using Theorem 2, it suffices to prove (1.2.1): then $S = \mathbb{N}$; any graph G with $\delta(G) \geq 2$ satisfies $|G| \in \mathbb{N} = S$; and G has a cycle by the definition of S.

Consider any $n \in \mathbb{N}$, and any graph $G = (V, E)$ with $|G| = n$ and $\delta(G) \geq 2$. Then $n = |G| \geq 3$. Thus we may assume $n - 1 \in S$.

Case 1: $\delta(G) \geq 3$. Let $v \in V$ and $G' = G - v$, i.e., form G' by deleting the vertex v together with all edges incident to it. Then $|G'| = n - 1$, and $\delta(G') \geq 2$, since no vertex lost more than one neighbor. Since $n - 1 \in S$, G' has a cycle, and this cycle is also a cycle of G.

Case 2: $\delta(G) = 2$. Pick $y \in V$ with $d(y) = 2$. Then $N(y) = \{x, z\}$ for distinct vertices x and z. If $xz \in E$ then G contains the cycle $xyzx$. Otherwise, let G' be the graph formed by deleting the vertex y, and the edges yx and yz, and adding the edge xz. Then $|G'| = n-1$ and $\delta(G') \geq 2$, since the loss of y is made up for by the fact that x and z are now new neighbors of each other. Since $n-1 \in S$, there is a cycle C contained in G'. If x and z do not appear consecutively in C then C is a cycle contained in G. If they do appear consecutively, we may assume that $C = xz \dots v_s x$. Then $xyz \dots v_s x$ is a cycle in G. □

What were the base and induction steps of this proof? The proof only applies to finite graphs, why? Can you construct an infinite graph with $\delta(G) \geq 2$ and no cycle?

The proof of Proposition 4 illustrates an important technique. Suppose we want to prove that all graphs have a property P, and f is a function on graphs (such as $|G|$) whose values are natural numbers. We can *argue by induction on* f as follows. First set $S = \{n \in \mathbb{N} :$ all graphs G with $f(G) = n$ have property $P\}$. Then argue by induction that $S = \mathbb{N}$. Now consider any graph G. As $f(G) \in \mathbb{N} = S$, the definition of S implies G has property P. Often the cleverness of a proof is in picking the *right* function f on which to do induction.

I have one last comment about a common mistake made by beginning students. When you argue by induction on f to prove a statement $P \to Q$ about graphs, the induction step starts with a big graph G that that satisfies P; then G is converted into a smaller graph G', i.e., $f(G) > f(G')$, that **also satisfies** P; you use the induction hypothesis to claim that G' satisfies Q; finally, you use that G' satisfies Q to prove that G satisfies Q. **You do not start with a small graph, and then make it bigger!**

1.3. Ramsey's Theorem for Graphs

Ramsey's Theorem is an important generalization of the Pigeonhole Principle. Here we only consider its simplest version applied to graphs; the general version is a statement about k-uniform hypergraphs. In the past it was presented as part of MAT 415, but it has been moved to MAT 416 because its presentation benefits from the language of graph theory.

Let $G = (V, E)$ be a graph, and suppose $X \subseteq V$. The set X is a *clique in* G if $xy \in E$ for all distinct vertices $x, y \in X$. It is an *independent set*, or *coclique*, in G if $xy \notin E$ for all vertices $x, y \in X$. A clique (coclique) X is a b-clique (b-coclique) if $|X| = b$. Let $\omega(G) := \max\{|X| : X \textit{ is a clique in } G\}$, *and* $\alpha(G) := \max\{|X| : X \textit{ is a coclique in } G\}$.

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is an *induced subgraph* of G if $H \subseteq G$ and $E(H) = \{xy \in E(G) : x \in V(H) \textit{ and } y \in V(H)\}$. For $X \subseteq V$, $G[X]$ is the induced subgraph of G that has vertex set X. The *complement* of G is the graph, $\overline{G} := (V(G), \overline{E}(G))$, where $\overline{E}(G) := \binom{V(G)}{2} \smallsetminus E(G)$.

Theorem 5 (Ramsey's Theorem 8.3.7,11). *For all graphs G and $a, b \in \mathbb{Z}^+$, if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.*

Proof. Argue by induction on $n = a + b$. (That is, let S be the set of natural numbers n such that for all positive integers a, b if $n = a + b$, and G is a graph with $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$. Show that for all $n \in \mathbb{N}$ if $\{0, \dots, n-1\} \subseteq S$ then $n \in S$.) Consider any $n = a + b$ with $a, b \in \mathbb{Z}^+$, and any graph G with $|G| \geq 2^{a+b-2}$.
Base step: $\min\{a, b\} = 1$. Since $|G| \geq 1$, G has a vertex v. Since $\{v\}$ is both a clique and an independent set, both $\omega(G) \geq 1$ and $\alpha(G) \geq 1$. So we are done regardless of whether $a = 1$ or $b = 1$.

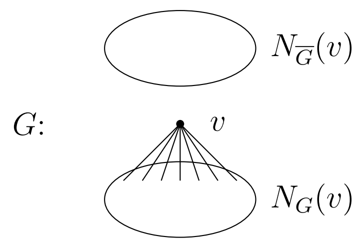

FIGURE 1.3.1. Ramsey's Theorem

Induction Step: $\min\{a,b\} \geq 2$ (so $a-1, b-1 \in \mathbb{Z}^+$). (We assume the induction hypothesis: the theorem holds for all $a', b' \in \mathbb{Z}^+$ with $a'+b' < a+b$.) Let $v \in V(G)$. Then

$$1 + d_G(v) + d_{\overline{G}}(v) = |G| \geq 2^{a+b-2} = 2^{a+b-3} + 2^{a+b-3}.$$

By the pigeonhole principle, either $d_G(v) \geq 2^{a+b-3}$ or $d_{\overline{G}}(v) \geq 2^{a+b-3}$.
Case 1: $d_G(v) \geq 2^{a+b-3}$. Set $H := G[N_G(v)]$. Then $|H| = d_G(v) \geq 2^{a-1+b-2}$. By the induction hypothesis H contains an $(a-1)$-clique X or a b-coclique Y. In the latter case Y is a b-coclique in G. In the former case $X+v$ is an a-clique in G.
Case 2: $d_{\overline{G}}(v) \geq 2^{a+b-3}$. Set $H := G[N_{\overline{G}}(v)]$. Then $|H| \geq 2^{a+b-1-2}$. By the induction hypothesis H contains an a-clique X or a $(b-1)$-coclique Y. In the former case, X is an a-clique in G. In the latter case $Y+v$ is a b-coclique in G. □

For $a, b \in \mathbb{Z}^+$, define $\text{Ram}(a,b)$ to be the least integer n such that every graph G with $|G| \geq n$ satisfies $\omega(G) \geq a$ or $\alpha(G) \geq b$. By Theorem 5, $\text{Ram}(a,b)$ exists and satisfies $\text{Ram}(a,b) \leq 2^{a+b-2}$.

HW 1. Prove that for $a, b \in \mathbb{Z}^+$ with $a, b \geq 2$:

(1) $\text{Ram}(a,b) = \text{Ram}(b,a)$.
(2) $\text{Ram}(a,1) = 1$.
(3) $\text{Ram}(a,2) = a$.
(4) $\text{Ram}(3,3) = 6$.
(5) $\text{Ram}(a,b) \leq \text{Ram}(a-1,b) + \text{Ram}(a,b-1)$.
(6) (+) $\text{Ram}(3,4) = 9$ (tricky, see Proposition 11).
(7) $\text{Ram}(4,4) \leq 18$ (use 6, even if you did not do it).
(8) (+) $\text{Ram}(4,4) = 18$.

It is known that $\text{Ram}(4,5) = 25$ and $43 \leq \text{Ram}(5,5) \leq 49$, and conjectured that 43 is the right answer. Proving it would make for a notable doctoral thesis.

THEOREM 6 (General Ramsey Theorem [**5**]). *For all $r, c, a_1, \ldots, a_c \in \mathbb{Z}^+$ there exists $n \in \mathbb{Z}^+$ such that for all sets V with $|V| \geq n$ and functions $f : \binom{V}{r} \to [c]$ there exist $i \in [c]$ and $H \subseteq V$ with $|H| = a_i$ such that $f(S) = i$ for all $S \in \binom{H}{r}$.*

Theorem 6 can be proved in the style of Theorem 5, but requires a "double induction" on r and then $\sum_{i=1}^{c} a_i$.

Our discussion of Ramsey's Theorem is just the tip of the iceberg. Whole books have been written on the subject.

1.4. Graph Isomorphism and the Reconstruction Conjecture

In order to study graph theory we need to know when two graphs are, for all practical purposes, the same.

DEFINITION 7. Two graphs G and H are *isomorphic* if there exists a bijection

$$f : V(G) \to V(H) \text{ such that } xy \in E(G) \text{ iff } f(x)f(y) \in E(H) \text{ for all } x, y \in V(G).$$

In this case we say that f is an isomorphism from G to H and write $G \cong H$. The isomorphism relation is an equivalence relation on the class of graphs. The equivalence classes of this relation are called *isomorphism types.* In graph theory we generally do not differentiate between two isomorphic graphs. We say that H is a *copy* of G to mean that $G \cong H$.

HW 2. Let $f : V(G) \to V(H)$ be an isomorphism between two graphs G and H. Prove carefully that $d_G(v) = d_H(f(v))$ for all $v \in V(G)$.

HW 3. Here we *do* distinguish between isomorphic graphs. Let $V = \{v, w, x, y, z\}$ be a set of five vertices, and $\mathcal{G} = \{G : G \text{ is a graph with } V(G) = V \text{ and } \|G\| = 4\}$. Determine (with proof) $|\mathcal{G}|$ and the number of isomorphism types of $\mathcal{G}$.

If x is a vertex of a graph G then $G - x$ is the induced subgraph $G[V(G) - x]$. The graph $G - x$ is called a vertex deleted subgraph of G.

DEFINITION 8. A *complete set of vertex deleted subgraphs* of a graph $G = (V, E)$ is a set $\mathcal{G}$ such that there exists a bijection $\psi : V \to \mathcal{G}$ with $\psi(x) \cong G - x$ for all $x \in V$. Notice that for distinct vertices x and y it may be that $G - x \cong G - y$. In this case $\psi(x)$ and $\psi(y)$ are *distinct copies* of $G - x$. See Figure 1.4.1. *A* complete set of vertex deleted subgraphs $\mathcal{G}$ of G is also called a *deck* of G, and the elements of $\mathcal{G}$ are called cards. Two graphs are *hypomorphic* if they have the same deck. See Figure 1.4.1.

Notice that G has infinitely many decks, but each deck $\mathcal{G}$ of G satisfies $|\mathcal{G}| = |G|$. We cannot determine $V(G)$ from its deck $\mathcal{G}$, but the following famous conjecture asks whether we can determine the isomorphism type of G from $\mathcal{G}$.

CONJECTURE 9 (Reconstruction Conjecture 1.3.12). *Any two hypomorphic graphs with at least three vertices are isomorphic.*

Now we take a little detour before proving Proposition 12

DEFINITION 10. An edge of a multigraph is called a link if it has two distinct ends, and a loop if both ends are the same vertex. The *degree* $d(v)$ of a vertex of a multigraph is the number of links incident to v plus twice the number of loops incident to v. In a multigraph there may be cycles of length 1—one loop—and cycles of length 2—two links between the same two vertices.

This definition is designed so that each edge is counted twice when you sum the degrees of a graph.

For a graph $G = (V, E)$ and pair $(v, e) \in V \times E$, set

$$\iota(v, e) := \begin{cases} 2 & \text{if } e \in E(v) \text{ is a loop} \\ 1 & \text{if } e \in E(v) \text{ is a link} \\ 0 & \text{otherwise} \end{cases}$$

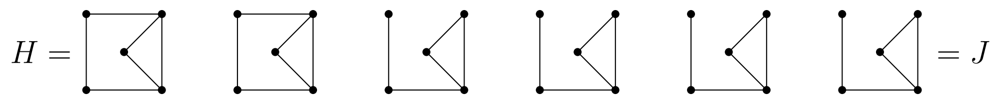

FIGURE 1.4.1. The vertex deleted subgraphs of a graph G. What is $|G|$? What is $\|G\|$? What is the isomorphism type of G?

PROPOSITION 11 (Handshaking 1.3.3.). *Every (multi-)graph $G := (V, E)$ satisfies $\sum_{v\in V} d(v) = 2\|G\|$. In particular, G has an even number of vertices with odd degree.*

PROOF.
$$\sum_{v\in V} d(v) = \sum_{v\in V}\sum_{e\in E} \iota(v, e) = \sum_{e\in E}\sum_{v\in V} \iota(v, e) = \sum_{e\in E} 2 = 2\|G\|. \qquad \square$$

PROPOSITION 12 (1.3.11). *For every graph $G = (V, E)$ with $|G| \geq 3$ and every $v \in V$,*
$$\|G\| = \frac{\sum_{v\in V} \|G - v\|}{|G| - 2} \qquad \text{and} \qquad d_G(v) = \|G\| - \|G - v\|.$$

PROOF. Every edge $e \in E$ satisfies $e \in E(G - v)$ if and only if $\iota(v, e) = 0$. Thus
$$\sum_{v\in V} \|G - v\| = \sum_{v\in V}\sum_{e\in E}(1 - \iota(v, e)) = \sum_{e\in E}\sum_{v\in V}(1 - \iota(v, e)) = \sum_{e\in E}(|G| - 2) = \|G\| (|G| - 2).$$
So the first equality holds. The second equality follows from $E = E(G - v) \cup E(v)$. $\square$

EXAMPLE 13. Suppose $\mathcal{G}$ is the deck consisting of 2 distinct copies of H and 4 distinct copies of J as shown in Figure 1.4.1. Find (with proof) a graph G such that if the deck of G' is $\mathcal{G}$ then $G' \cong G$.

SOLUTION. Let $G = (V, E)$ be an arbitrary graph for which $\mathcal{G}$ is a deck. Then $|G| = |\mathcal{G}| = 6$. Using Proposition 12, we have:
$$\|G\| = (2\|H\| + 4\|J\|)/(|G| - 2) = (2 \cdot 6 + 4 \cdot 5)/4 = 8.$$
Consider $x \in V$ with $G - x \cong H$. Then $d_G(x) = 8 - \|H\| = 2$. By inspection, $G - x$ has a vertex z with two adjacent neighbors w_1, w_2 on a 4-cycle $w_1w_2w_3w_4w_1$. As J does not contain a 4-cycle, $G - z \not\cong J$, and so $G - z \cong H$. Thus $d(z) = 2$. As $\Delta(G) = 3$, we already know $N(z) = \{w_1, w_2\}$, $N(w_1) = \{w_2, w_4, z\}$, and $N(w_2) = \{w_2, w_3, z\}$. So there are only two possibilities for the two neighbors of x, and $N(x) = \{w_3, w_4\}$. See Figure 1.4.3. $\square$

HW 4. (*) The *degree sequence* of a graph $G = (V, E)$ is a nondecreasing sequence of integers $d_1, \ldots, d_{|G|}$ such that $d_i = d(v_i)$ for all $i \in [|G|]$ and $V = \{v_1, \ldots v_{|G|}\}$. For example the degree sequence for the graph in Figure 1.4.3 is $2, 2, 3, 3, 3, 3$. Give a (small) example (with proof) of two graphs that have the same degree sequence, but are not isomorphic.

HW 5. (*) Find a graph G such that the graphs shown in Figure 1.4.2 form a deck of $\mathcal{G}$. Prove that if $\mathcal{G}$ is also the deck of H then $G \cong H$.

HW 6. (*) A graph is *regular* if all its vertices have the same degree. Prove that if two regular graphs have the same deck then they are isomorphic.

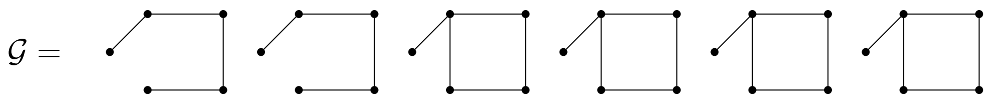

FIGURE 1.4.2. HW 5

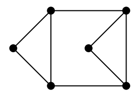

FIGURE 1.4.3. Discovered graph G

1.5. Some Important Graphs and Graph Constructions

A *path* is a graph $P = (V, E)$ such that V can be ordered as $v_1, \dots, v_{|P|}$ so that $E = \{v_i v_{i+1} : i \in [|P| - 1]\}$. The *length* of the path P is $\|P\|$. A path with only one vertex is possible; such paths are said to be *trivial.* Clearly, any two paths with the same length are isomorphic (HW 10). We use the notation P_n to denote a fixed path of length $n - 1$. Then if P is a path of length $n - 1$, we say that P is a copy of P_n, or more carelessly $P = P_n$. We write $v_1 v_2 \dots v_n$ (without commas) to denote a copy of P_n whose edge set is $\{v_i v_{i+1} : i \in [n - 1]\}$. For a path $P = v_1 \dots v_n$ set $v_i P = v_i \dots v_n$, $v_i P v_j = v_i \dots v_j$, and $P v_j = v_1 \dots v_j$. The vertices v_1 and v_n are called the *ends* of P_n. The other vertices are *internal* vertices. Suppose G is a graph and $X \subseteq V(G)$. A path $P \subseteq G$ is an X-path if its ends, but not its internal vertices, are contained in X. An $\{x, y\}$-path is usually called an x, y-path. The *distance* $d(x, y)$ between x and y is the length of the shortest x, y-path.

A *cycle* is a graph C formed by adding the additional edge $v_1 v_n$ to a path $v_1 v_2 \dots v_n$ with $n \geq 3$. Again, the length of C is $\|C\|$. Clearly any two cycles with the same length are isomorphic. We use the notation C_n for a fixed cycle of length n. We write $v_1 v_2 \dots v_n v_1$ to denote a copy of C_n whose edge set is $\{v_i v_{i \oplus 1} : i \in [n]\}$, where $\oplus$ denotes addition modulo n. The *girth* of a graph G is the length of its shortest cycle C with $C \subseteq G$, if there is one; otherwise the *girth* is infinity. The *circumference* of G is the length of the longest cycle C with $C \subseteq G$, if there is one. Otherwise the circumference is zero.

A *complete graph* is a graph $K = (V, E)$ such that $xy \in E$ for all $x, y \in V$. We use the notation K_n for a fixed complete graph with n vertices. Notice that the vertices of a complete graph are a clique. Then $\overline{K}_n$ is a graph with n vertices and no edges, and the vertices of $\overline{K}_n$ are a coclique. We call $\overline{K}_n$ the *empty graph on n vertices.* Now we introduce some notation not in the text: $K(A, B)$ denotes the graph (V, E) such that $V = A \cup B$ and $E = \{ab : a \neq b \wedge (a, b) \in A \times B\}$. Then $K(A, A)$ denotes a complete graph whose vertex set is A; we abbreviate this by $K(A)$. Finally, for $a, b \in \mathbb{Z}^+$, let $K_{a,b}$ denote a graph of the form $K(A, B)$, where $|A| = a$, $|B| = b$, and $A \cap B = \emptyset$. Such a graph is called a *complete bipartite graph.* (We will have more to say about bipartite graphs shortly.)

Let $G = (V, E)$ and $H = (W, F)$ be graphs. Define the *sum* of G and H by

$$G + H := (V \cup W), E \cup F)$$

and the *join* of G and H by

$$G \vee H := (G + H) + K(V, W).$$

The k-th power of G is the graph $G^k = (V, E^k)$, where $E^k = \{uv : d(u, v) \leq k\}$.

The *Petersen* graph has the form $(\binom{[5]}{2}, \{AB : A \cap B = \emptyset, A, B \in \binom{[5]}{2}\})$.

HW 7. (*) Prove that every graph G with $|G| < \|G\|$ contains a P_4.

Definition 14. A decomposition of a graph G is a set of subgraphs such that each edge of G appears in exactly one subgraph of the set.

Example 15. K_4 can be decomposed into two P_4's; it can also be decomposed into three P_3's, and into $K_3, K_{1,3}$.

HW 8. Prove that K_n decomposes into three isomorphic subgraphs if and only if $n+1$ is not divisible by 3.

1.6. Connection in graphs

Definition 16 (1.2.2). A *walk* $W = (V, E, \sigma)$ is a graph (V, E) and a sequence $\sigma = (v_1, \dots, v_n)$ of all the vertices in V with possible repetitions such that $v_i v_{i+1}$ is an edge for each $i \in [n-1]$. Let $G(W) = (V, E)$, $V(W) = V$, and $E(W) = W$. We denote the walk W by $v_1 \dots v_n$. If the vertices of σ are distinct then this notation also denotes a path. If $v_1 = v_n$ then W is *closed*; otherwise it is *open*. If W is open then v_1 and v_n are its *ends*, and all other vertices are *internal*. If all the edges $v_i v_{i+1}$ are distinct then W is a *trail*. The *length* of W is $n-1$. The walk W is a u, v-walk (trail, path) if $u = v_1$ and $v = v_n$. The trivial walk v_1 of length 0 is closed. We say that W contains a walk W', or W' is a subwalk of W, if there exists a subsequence $W' = v_{i_1} v_{i_2} \dots v_{i_s}$ such that W' is a walk, and each edge $v_{i_h} v_{i_{h+1}}$ of W' has the form $v_j v_{j+1}$ for some $j \in [n-1]$. W is a walk in a graph G if $V(W) \subseteq V(G)$ and $E(W) \subseteq E(G)$. We use the notation Wv_j, $v_i W v_j$, and $v_i W$ to indicate the *subwalks* $v_1 \dots v_j$, $v_i \dots v_j$, *and* $v_i \dots v_n$. *Also* $W^* := v_n v_{v-1} \dots v_1$.

Lemma 17 (1.2.5). *Every u, v-walk $W = v_1 \dots v_n$ contains a u, v-path.*

Proof. We argue by induction on the length l of W. If W is a path then we are done. Otherwise, there exist $i, j \in [n]$ with $i < j$ such that $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1}$ and $W' = v_1 \dots v_i v_{j+1} \dots v_n$ is a shorter u, v-walk contained in W. By induction W' contains a u, v-path P, and P is also contained in W. □

Definition 18 (1.2.6). Let $G = (V, E)$ be a graph. Vertices $x, y \in V$ are *connected* (regardless of whether $xy \in E$) if there is a u, v-walk in G. The graph G is *connected* if all vertices x and y are connected. The *connection relation* is the set of ordered pairs (u, v) of G with u and v connected.

Proposition 19. *The connection relation is an equivalence relation.*

HW 9. (-) Let G be a graph with $x, y, z \in V(G)$. Prove that if G contains an x, y-path and a y, z-path then it contains an x, z-path. Be careful; it is not completely trivial.

Definition 20 (1.2.8). A *component* of G is a subgraph $H = G[X]$ induced by an equivalence class X of the connection relation.

HW 10. (+) Prove that if two connected graphs G and H on n vertices both have degree sequence $1, 1, 2, \dots, 2$ (two ones followed by $n-2$ twos) then they are isomorphic.

HW 11. (*) Let $P \subseteq G$ be an x, y-path. Prove that $G[P]$ contains an x, y-path Q with $Q = G[Q]$.

HW 12. (*) Prove that any two paths P and Q with maximum length in a connected graph have a common vertex.

1.7. Bipartite graphs

DEFINITION 21. Consider a graph $G = (V, E)$ and sets $A, B \subseteq V$. Let $E(A, B)$ denote the set of edges with one end in A and one end in B, and abbreviate $E(\{x\}, B)$ as $E(x, B)$. Let $\|x, B\| = |E(x, B)|$ and $\|A, B\| = \sum_{a \in A} |E(a, B)|$, that is the number of edges in $E(A, B)$ counting those edges with both ends in A twice. A graph $G = (V, E)$ is *bipartite* if it has a *bipartition*, that is a partition of V into one or two independent sets. This means that $E = \emptyset$ or there exists a partition $\{A, B\}$ of V (with $V = A \cup B$, $A \cap B = \emptyset$) such that both A and B are independent, or equivalently, $E = E(A, B)$. An A, B-*bigraph* is a bipartite graph with bipartition $\{A, B\}$. Notice that a graph is bipartite if and only if it is a subgraph of a complete bipartite graph.

Many theorems in graph theory assert the existence of some special structure in a graph—say a bipartition. To show that a particular graph has such a structure it is enough to make a lucky guess, and check that your guess provides the structure. In general, it is much harder to show that a graph does not have the desired structure. Typically this would require an exhaustive search of exponentially many possibilities—say all $2^{|G|}$ partitions of the vertices into at most two parts. However for some structures we can prove the existence of *obstructions* with the property that every graph either has the structure or it has an obstruction, but not both. In this case, a lucky guess of an obstruction provides a proof that the structure does not exist. Theorem 24 is an example of this phenomenon.

DEFINITION 22. A path, cycle, trail, walk W is *even* (*odd*) if its length is even (odd).

LEMMA 23. *Every odd closed walk $W = v_1 \dots v_n v_1$ contains an odd cycle.*

PROOF. Argue by induction on the length of W. If W is a cycle we are done. Otherwise, as $\|W\| \geq 3$, there exist integers $1 \leq i < j \leq n$ with $v_i = v_j$. Then $v_i v_{j+1} = v_j v_{j+1} \in E$ and $W' := v_1 \dots v_i v_{j+1} \dots v_n v_1$ and $W'' := v_i v_{i+1} \dots v_j$ are shorter closed walks, whose lengths sum to the length of W. So one of them must be odd. By the induction hypothesis, the odd one contains an odd cycle, which is also contained in W. □

Observe that if $xy \in E$ then xyx is an even closed walk that does not contain any cycle.

THEOREM 24 (1.2.18). *A graph $G = (V, E)$ is bipartite iff it contains no odd cycle.*

PROOF. First suppose G is bipartite with bipartition $\{A, B\}$. It suffices to show that if $C \subseteq G$ is a cycle then it is even. Since G is bipartite, $E(C) \subseteq E \subseteq E(A, B)$. So each edge $e \in E(C)$ has exactly one end in A. Thus the length of C is the even number

$$\|C\| = \sum_{e \in E(C)} \sum_{v \in A \cap V(C)} \iota(v, e) = \sum_{v \in A \cap V(C)} \sum_{e \in E(C)} \iota(v, e) = \sum_{v \in A \cap V(C)} d_C(v) = 2|A \cap V(C)|.$$

Now suppose G is not bipartite. Then some component $H \subseteq G$ is not bipartite (why?). Let $x \in V(H)$. Set

$$A := \{v \in V(H) : \text{ there exists an odd } x, v\text{-walk in } H\} \text{ and}$$
$$B := \{v \in V(H) \smallsetminus A : \text{ there exists an even } x, v\text{-walk in } H\}.$$

Since H is a component of G, it is connected, so $A \cup B = V(H)$; by definition $A \cap B = \emptyset$. As H is not bipartite there is an edge $uv \in E(H[A]) \cup E(H[B])$. Thus there are walks xPu and xQv with the same parity (odd if $u, v \in A$, even if $u, v \in B$). So $W = xPuvQ^*x$ is an odd closed walk. By Lemma 23 there is an odd cycle $C \subseteq W$, and $C \subseteq H \subseteq G$. □

HW 13. (*) Prove that a 3-regular graph G decomposes into $K_{1,3}$'s if and only if G is bipartite.

HW 14. (*) Prove that a graph G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}|H|$ for all $H \subseteq G$.

1.8. Dirac's Theorem

Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. An *A, B-walk* is a walk whose first vertex is in A, whose last vertex is in B and whose interior vertices are in neither A nor B. If $A = \{a\}$ or $B = \{b\}$, we may shorten this notation to an a, B-walk or an A, b-walk. Similarly, if $A = B$, we may shorten it to a B-walk. Also, if $H \subseteq G$ an H-walk is a $V(H)$-walk, etc.

A path in a graph G is *maximal* if it is not a subpath of a longer path in G. It is *maximum* if there is no longer path in G.

Recall that the *minimum degree* of G is $\delta(G) = \min\{d(v) : v \in V\}$. Similarly, the *maximum degree* of G is $\Delta(G) = \max\{d(v) : v \in V\}$. If $\delta(G) = r = \Delta(G)$ then G is *r-regular*.

An *embedding* of H into G is an isomorphism from H to a subgraph of G. If there exists an embedding of H in G then we say that H can be *embedded* in G, or that H is embeddable in G. A subgraph $H \subseteq G$ is said to be a *spanning* subgraph of G if $V(H) = V(G)$. A spanning cycle of G is called a *hamiltonian* cycle. If G contains a hamiltonian cycle, G is said to be *hamiltonian*.

Many questions in graph theory have the following form: Given two graphs G and H with $|H| = |G|$ what "local" conditions on G ensure that H is embeddable in G? If G is complete then trivially H is embeddable in G. This is guaranteed by the local condition $\delta(G) = |G| - 1$, but in many cases we can do much better. Corollaries 26 and 27 below are examples.

We have seen that not only can the question of whether a graph is bipartite be answered positively with proof by a luck guess—the bipartition, it can also be answered negatively with proof by a lucky guess—an odd cycle. The question of whether a graph is hamiltonian can also be answered positively with proof by a lucky guess—the hamiltonian cycle. However there is no "efficient" guessing method known for deciding with proof that a graph has no hamiltonian cycle, and it is strongly believed that there is no such method.

Intuitively, if a graph has enough edges—for instance if it is complete—then it is hamiltonian. Here are some ways of quantifying what "enough" means.

THEOREM 25. *Every connected graph $G = (V, E)$ with $|G| \geq 3$ contains a path or cycle of length at least $l = \min\{|G|, d(x) + d(y) : xy \notin E\}$.*

PROOF. Let $P = v_1 \dots v_t$ be a maximum path in G. If $\|P\| \geq l$ then we are done. So assume $\|P\| \leq l - 1$, i.e. $t \leq l$. We first prove:

(1.8.1) $$G \text{ contains a cycle with } V(C) = V(P).$$

If G is complete then, $t = |G| \geq 3$. Else G has two nonadjacent vertices x and y. As G is connected, there is an x, y-path Q. Then $t \geq \|Q\| + 1 \geq 3$. Anyway $t \geq 3$. If $v_1 v_t \in E$ then,

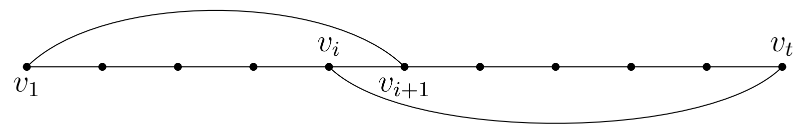

FIGURE 1.8.1. Hamiltonian cycle

as $t \geq 3$, $C := v_1 P v_t v_1$ is the desired cycle. Otherwise, $v_1 v_t \notin E$. So

$$t \leq l \leq d(v_1) + d(v_t). \tag{1.8.2}$$

Since P is maximum, $N(v_1), N(v_t) \subseteq P$. Let

$$X = \{i \in [t-1] : v_1 v_{i+1} \in E\} \text{ and } Y = \{i \in [t-1] : v_t v_i \in E\}.$$

Then $|X| = d(v_1)$ and $|Y| = d(v_t)$. By (1.8.2) and inclusion-exclusion:

$$t - 1 \geq |X \cup Y| = |X| + |Y| - |X \cap Y| \geq t - |X \cap Y|$$
$$|X \cap Y| \geq 1.$$

Let $i \in X \cap Y$ (Figure 1.8.1). Then the cycle $C = v_1 v_{i+1} P v_t v_i P v_1$ spans P, proving (1.8.1).

Since $l \leq |G|$ and $|C| = \|C\|$, it suffices to show $|G| = |C|$. If not, there is $x \in V \smallsetminus V(C)$. Since G is connected, there is an x, C-path $Q = x \dots u_1$. Choose notation so that $C = u_1 \dots u_t u_1$. Then $P' = xQu_1 u_2 \dots u_t$ is a longer path than P, a contradiction. □

COROLLARY 26 (Dirac's Theorem 1952 **[2]** 7.2.8). *If $\delta(G) \geq \frac{1}{2}|G| > 1$ then G is hamiltonian.*

PROOF. By Theorem 25, it suffices to prove G is connected. Consider $x, y \in V(G)$. Then

$$|G| \geq |N[x] \cup N[y]| = |N[x]| + |N[y]| - |N[x] \cap N[y]| \geq |G| + 2 - |N[x] \cap N[y]|$$
$$|N[x] \cap N[y]| \geq 2.$$

So x is connected to y by a path of length at most 2. □

Here is a weaker, but slightly less local condition that ensures a graph is hamiltonian.

COROLLARY 27 (Ore's Theorem 1960 7.2.9). *If G is a graph with $|G| \geq 3$ and $d(x) + d(y) \geq |G|$ for all distinct nonadjacent vertices x and y then G is hamiltonian.*

PROOF. As in the proof of Corollary 26, G is connected, since for distinct vertices x and y either $xy \in E$ or $|N[x]| + |N[y]| \geq |G| + 2$. So we are done by Theorem 25. □

Pósa (when he was in high school) posed a conjecture extending Dirac's Theorem:

CONJECTURE 28 (Pósa 1963). *If $\delta(G) \geq \frac{2}{3}|G|$ then G contains the square (2-power) of a hamiltonian cycle.*

The next Theorem answers a related question.

THEOREM 29 (Fan & Kierstead 1996 **[3]**). *If $\delta(G) \geq \frac{2|G|-1}{3}$ then G contains the square of a hamiltonian path. The degree condition is best possible.*

The next conjecture generalizes Corollary 26 and Conjecture 28.

Conjecture 30 (Seymour 1974). *If $\delta(G) \geq \frac{k}{k+1}|G|$ then G contains the k-th power of a hamiltonian cycle.*

Seymour's Conjecture was proved for sufficiently large graphs (even more vertices than the number of electrons in the known universe when $k \geq 2$).

Theorem 31 (Komlós, Sárközy & Szemerédi 1998 [**4**]). *For every integer k there exists an integer n such that Conjecture 30 is true for graphs G with $|G| \geq n$.*

The next theorem improves the bound on n when $k = 2$.

Theorem 32 (Châu, DeBiasio & Kierstead 2011 [**1**]). *Conjecture 28 is true for graphs G with $|G| \geq 2 \times 10^8$.*

HW 15. Prove that $K_{a,a-1}$ is not hamiltonian for any integer $a > 1$. More generally, prove that if $\alpha(G) > \frac{1}{2}|G|$ then G is not hamiltonian. Determine $\delta(K_{a,a-1})$.

HW 16. For disjoint sets $A, B, \{v\}$, let $G = \overline{K}(A, B) + K(v, A \cup B)$. Prove that G is not hamiltonian. Determine $\delta(G)$ when $|A| = |B|$.

HW 17. Let $P = v_1 \dots v_t \subseteq G$ be a maximum path and $x \in V(G - P)$. Prove that $\|\{v_t, x\}, P\| \leq t - 1$.

HW 18. Prove that if $\delta(G) \geq \frac{|G|-1}{2}$ then G has a hamiltonian path.

HW 19. $(+)$ Let G be an X, Y-bigraph with $|X| = |Y| = k$ and $\delta(G) \geq \frac{k+1}{2} \geq 2$. Prove that G contains a hamiltonian cycle. [Hint: First prove that if G contains a maximal path $P = P_t$ then $G[P]$ contains a cycle of length at least $2\lfloor \frac{t}{2} \rfloor$. Then show that G contains a hamiltonian path.] For all k give an example to show that the bound on the minimum degree cannot be lowered.

1.9. Even graphs and Euler's Theorem

Sometimes when proving a statement by induction it is easier to prove a stronger statement. This phenomenon is called the *inventors paradox.* The reason this is possible is that while more must be proved, the induction hypothesis provides more to base an argument on. In this section we see an elementary example of this. It is easier to prove our result for multigraphs. Proposition 35 is a minor example of this; we will see more serious examples later.

Definition 33. A multigraph is *eulerian* if it has a closed trail containing all edges. (Note that $T = v$ is closed, since its only vertex is its first and its last.) Such a trail is said to be an eulerian trail. A multigraph is *even if every vertex has even degree.*

For $H \subseteq G$ and $v \in V(G) \setminus V(H)$, set $d_H(v) := 0$. If T is a trail in a graph G then let $d_T(v) = d_{G(T)}(v)$.

Fact 34. *If H and G are even graphs with $H \subseteq G$ then $H' := G - E(H)$ is even.*

Proof. Since G and H are even, every $v \in V(G)$ satisfies

$$d_{H'}(v) = d_G(v) - d_H(v) \equiv 0 \mod 2.$$

□

Proposition 35. *Let $T = v_1 ... v_n$ be a trail in a multigraph $G = (V, E)$. Then $d_T(v)$ is even for every vertex v, except that if T is open then $d_T(v_1)$ and $d(v_n)$ are odd.*

Proof. Let $T' := T + v_n v_1$ and $G' = G + v_n v_1$ ($v_n v_1$ may be a loop of multiple edge). Then T' is closed. It suffices to show that $d_{T'}(v_i)$ is even for all $v \in V$, since $d_T(v_i) \not\equiv d_{T'}(v_i)$ mod 2 if and only if $v_i \in \{v_1, v_n\}$ and $v_1 \neq v_n$. Argue by induction on n.

If T' is a cycle, or $t = 1$, then $d_{T'}(v_i) \in \{0, 2\}$ for every $v \in V(G)$. Otherwise, there exist $1 < i < j \leq t$ with $v_i = v_j$. Let $T_1 = v_1 T' v_i v_{j+1} T' v_1$ and $T_2 = v_i T' v_j (= v_i)$. Then T_1 and T_2 are both closed trails, shorter than T', and every edge of T' is in exactly one of T_1 and T_2. By the induction applied to T_1 and T_2,

$$d_{T'}(v) = d_{T_1}(v) + d_{T_2}(v) \equiv 0 + 0 \equiv 0 \mod 2.$$

□

Theorem 36 (Euler (1736) 1.2.26). *A multigraph G is eulerian iff it has at most one nontrivial component and it is even.*

Proof. First suppose G has a Eulerian trail T. Since T is connected it only contains edges from one component. Since T contains all edges, G has only one nontrivial component. Since T is closed and contains all edges of G, Proposition 35 implies every vertex of G has even degree (possibly 0).

Now suppose G has at most one nontrivial component H, $v \in V(H)$, and every vertex has even degree. Let $T = v_1 \dots v_t$ be a maximum length trail in G; it exists because (v) is a candidate and $|G| < \infty$. Then T is closed: Otherwise v_t is incident to an odd number of edges of T by Proposition 35. Since $d(v_t)$ is even it is incident to some edge $v_t v$ that is not in T. So we can extend T to $T^+ = v_1 T v_t v$, contradicting the maximality of T.

It remains to show that $E(H) \subseteq E(T)$. Otherwise there is an edge $ab \in E(H) \smallsetminus E(T)$. Since H is connected there is an $\{a, b\}, V(T)$-path P. Choose notation so that $P = b \dots v_i$. By definition, $a \notin V(P)$ (but maybe $b = v_i$). Since T is closed, $T^+ = abPv_i T v_i$ is a longer trail than T, a contradiction. □

Lemma 37. *Every graph G with $\delta(G) \geq 2$ contains a cycle.*

Proof. Let $P = v_1 \dots v_t$ be a maximum path in G; it exists since $1 \leq |G| \leq \infty$. Then $N(v_t) \subseteq V(P)$. So there exist $i < t - 1$ such that $v_t v_i \in E(G)$. Thus $v_i P v_t v_i$ is a cycle contained in G. □

Corollary 38 (1.2.25). *If G is an even graph with $\|G\| > 0$ then G contains a cycle.*

Proof. Some component H of G contains an edge. Since H is connected, $\delta(H) \geq 1$. Since G is even this can be strengthened to $\delta(H) \geq 2$. So by Lemma 37, $H \subseteq G$ contains a cycle. □

Second proof of Theorem 36 (Sufficiency). Suppose G is even and has at most one nontrivial component G'. We argue by induction on $\|G'\|$. If G' is a cycle (or $\|G'\| = 0$), then the cycle in its natural order (or any vertex) is the Eulerian trail.

Otherwise, by Corollary 38, G' contains a cycle C. Let H be a nontrivial component of $G' - E(C)$ (maybe $H = G' - E(C)$), and set $H' = G' - E(H)$. Both H and H' are even. Also H' is connected, since all components of $G' - E(C)$ that are contained in H' are connected to each other in H' by edges of C. Moreover, $\|H\| \leq \|G'\| - \|C\| < \|G'\|$

and $\|H'\| = \|G'\| - \|H\| < \|G'\|$. So H and H' are nonempty even connected graphs with $\|G'\| = \|H\| + \|H'\|$. By the induction hypothesis H and H' contain Eulerian trails T and T'. Moreover, T contains a vertex $v_1 \in C$ and T' contains all vertices of C. Choose notation so that $T = v_1 \dots v_n v_1$ and $T' = v_1 u_2 \dots u_m v_1$. Then $v_1 T v_n v_1 T' u_m v_1$ is an Eulerian trail in G', and G. □

THEOREM 39 (1.2.33). *A connected graph G with exactly q vertices of odd degree decomposes into* $\max\{1, \frac{q}{2}\}$ *trails.*

PROOF. By Lemma 11, q is even. Let G^+ be the result of adding a new vertex v^+ to G so that $N(v^+)$ is the set of vertices with odd degree in G. Since q is even, and every $v \in V(G)$ satisfies $d_{G^+}(v) \equiv d_G(v) + 1 \mod 2$ if and only if $d_G(v)$ is odd, G^+ is even. By Theorem 36, G^+ has an Eulerian Trail T. Removing v^+ partitions T into $\frac{q}{2}$ trails that decompose G. □

Alternatively, we could have proved Theorem 39 by adding q edges connecting disjoint pairs of odd degree vertices.

CHAPTER 2

Cut-vertices, -edges and trees

Definition 40 (1.2.12.). A *cut-vertex* is a vertex in a graph G is a vertex such that $G - v$ has more components than G. Similarly, a *cut-edge* is an edge of G such that $G - e$ has more components than G.

Notice that a vertex v is a cut-vertex of G if and only if $H - v$ is neither empty nor connected, where H is the component of G (maybe $H = G$) containing v. Similarly an edge e is a cut-edge of G if and only if $H - e$ is not connected, where H is the component of G containing e. Thus G is not a cut-edge if its ends are connected in $G - e$.

Theorem 41 (1.2.14.). *An edge $e = xy$ in G is not a cut-edge iff it belongs to a cycle.*

Proof. First suppose e is not a cut edge. Then there exists an x, y-path P in $G - e$. So $xPyx$ is a cycle in G. Now suppose e is on a cycle C in G. Then $x(C - e)y$ is a path connecting the ends of e, and so e is not a cut-edge. □

Theorem 42. *The ends of a maximal path $P = x \dots y \subseteq G$ are not cut-vertices of G.*

Proof. Let H be the component of G containing x. Suppose x is a cut-vertex. Then $|H| \geq 2$, $y \neq x$. and some $u \in V(H - x)$ is not connected to y. As H is connected, there is a u, y walk W in H. If $x \notin V(W)$ then set $W' = W$; else the predecessor v of x on W is in $N(x) \subseteq P - x$; set $W' = uWvPy$. Anyway W' is a u, y-walk, a contradiction. □

Definition 43. A graph is *acyclic* if it contains no cycle. Acyclic graphs are also called *forests*. A connected acyclic graph is called a *tree*. A *leaf* is a vertex v with $d(v) = 1$. We say that a graph G satisfies (A) if it is acyclic, (C) if it is connected, and (E), if $|G| = \|G\| + 1$.

Lemma 44. *A graph G with $\|G\| \geq 1$ has at least two leaves if it satisfies (A) or both (C) and (E).*

Proof. First suppose that G is acyclic. Let $P = v_1 \dots v_t$ be a maximum path in G. Since G has an edge, $v_1 \neq v_t$. Since P is maximum and acyclic $N(v_1) = \{v_2\}$ and $N(v_t) = \{v_{t-1}\}$. So v_1 and v_t are distinct leaves.

Now suppose that G satisfies (C) and (E). Let L be the set of leaves in G. Since G is connected and has an edge, $\delta(G) \geq 1$. Since G satisfies (E),

$$\begin{aligned} 2|G| - |L| &\leq \sum_{v \in V(G)} d(v) = 2\|G\| =_{(E)} 2|G| - 2 \\ 2 &\leq |L|. \end{aligned}$$ □

Lemma 45. *Suppose G is a graph with a leaf l and $G' = G - l$. Then each condition (A), (C), (E) is satisfied by G iff it is satisfied by G'.*

Proof. Suppose G is acyclic. Since removing a vertex cannot create a cycle, G' is acyclic. Now suppose G' is acyclic. Since every vertex in a cycle has degree 2, adding a leaf l cannot create a cycle, and so G is acyclic.

Suppose G' is connected. Since l has a neighbor in $V(G')$, G is connected. Now suppose G is connected. Since $d(l) = 1$, there is a maximal path P with an end l. Thus l is not a cut-vertex, and so G' is connected.

Since $|G| = |G'| + 1$ and $\|G\| = \|G'\| + 1$, G satisfies (E) iff G' does. □

Theorem 46 (2.1.4). *If a graph G satisfies at least two of the conditions (A), (C), and (E) then it satisfies all three.*

Proof. Argue by induction on $|G|$.
Base Step: $|G| = 1$. By inspection, G satisfies all of (A), (C) and (E).
Induction Step: $|G| \geq 2$. Since G satisfies (C) or (E), it has an edge. Since G satisfies (A) or both (C) and (E), it has a leaf l by Lemma 44. Let $G' = G - l$. By Lemma 45, G' satisfies the same two conditions that G does. By the induction hypothesis G' satisfies all three conditions. By Lemma 45, G does also. □

For emphasis, we state that Theorem 46 implies every tree T satisfies $|T| = \|T\| + 1$.

Theorem 47 (2.1.4). *G is a tree iff there is exactly one path between any two vertices.*

Proof. Both statements imply G is connected. So it suffices to show that if G is connected, then G has a cycle C if and only if there are vertices x and y and distinct x, y-paths P and Q. If C is a cycle with edge xy then let $P = xy$ and $Q = x(C - xy)y$. If P and Q are distinct x, y-paths with (say) $uv \notin E$ then uP^*xQyP^*v is a u, v-walk in $G - uv$, so uv is not a cut-edge; by Theorem 41, G has a cycle containing xy. □

A *spanning tree* of a graph G is a tree $T \subseteq G$ with $V(T) = V(G)$.

Corollary 48 (2.1.5). *Let $T = (V, E)$ be a tree. Then*

(1) Removing an edge disconnects T.
(2) Adding an edge $xy \notin E$ with $x, y \in V$ to T creates a unique cycle.

Proof. (1) Since T is acyclic, every edge is a cut edge by Theorem 41.

(2) By Theorem 47 there is a *unique* x, y-path $P \subseteq T$. So $P + xy$ is a cycle in $T' := T + xy$. Consider any cycle $D \subseteq T'$. Since T is acyclic, $xy \in E(D)$. By the uniqueness of P, $D - xy = P$; so $D = P + xy$. □

Corollary 49. *Let $G = (V, E)$ be a connected graph.*

(1) Every minimally connected spanning subgraph $F \subseteq G$ is acyclic.
(2) Every maximally acyclic spanning subgraph $F \subseteq G$ is connected.
(3) Every maximal subtree $F \subseteq G$ is spanning.

Proof. (1) Let F be a minimal connected, spanning subgraph of G. Then every edge of F is a cut-edge. By Theorem 41, no edge of F is on a cycle, so F is acyclic.

(2) Let F be a maximal acyclic spanning subgraph of G. First consider any edge $xy \in E$. If $xy \in E(F)$ then x and y are connected in F; else by maximality there is a cycle $C \subseteq F + xy$ with $xy \in E(C)$, and so x and y are connected by the path $x(C - xy)y \subseteq F$. Now consider any $u, v \in V$. We show by induction on $t := \text{dist}_G(u, v)$ that u is connected to v in F. We

have already seen this if $t \leq 1$. If $t \geq 2$ then let $(u =)v_1 \dots v_{t-1}v_t(= v)$ be a u, v-path in G. By induction u is connected to v_{t-1} and v_{t-1} is connected to v in F, so we are done.

(3) Let $F \subseteq G$ be a subtree with $|F|$ maximum. Consider any $x \in V(G)$. Since G is connected, there is an x, F-path $(x =)x_1 \dots x_t \subseteq G$. As $|F|$ is maximum, $t = 1$, and $x \in V(F)$. □

PROPOSITION 50 (2.1.5). *Suppose T and T' are spanning trees of a graph G. Then for every $e = ab \in E(T) \setminus E(T')$*

(1) there exists $e' \in E(T') \setminus E(T)$ such that $T - e + e'$ is a spanning tree of G; and
(2) there exists $e' \in E(T') \setminus E(T)$ such that $T' + e - e'$ is a spanning tree of G.

PROOF. (1) Let $H = T \cup T' - e$. Then H is a connected, spanning subgraph of G, $T^- := T - e$ is acyclic, and using Theorem 46, $|T^-| = |T| = \|T\| + 1 = \|T^-\| + 2$ and T^- is not a tree. By Corollary 49(2), T^- is not maximally acyclic in H. So there is $e' \in E(H - T^-) = E(T')$ with $T^* := T^- + e'$ acyclic. As $|T^*| = \|T^*\| + 1$, Theorem 46 implies $T^* = T - e + e'$ is a tree, spanning G.

(2) By Corollary 48(2), $T' + e$ contains a unique cycle C. Since T is acyclic, $C \nsubseteq T$; let $e' \in E(C - T) \subseteq E(T')$. Then $T^* := T' + e - e'$ is acyclic, since e' is an edge of the unique cycle in $T' + e$. By Theorem 46, T^* is a tree, since $|T^*| = |T| = \|T\| + 1 = \|T^*\| + 1$. □

DEFINITION 51. If $G = (V, E)$ is a graph with a function $w : E \to \mathbb{N}$ then w is called a *weight function*. For $H \subseteq G$ set

$$w(H) = \sum_{e \in E(H)} w(e).$$

A *minimal spanning tree* of G (with respect to w) is a spanning tree $T \subseteq G$ such that $w(T) \leq w(T')$ for all spanning trees $T' \subseteq G$. Consider the following algorithm.

Minimal Spanning Tree Algorithm:

(1) input a connected graph $G = (V, E)$ with a weight function w;
(2) let $e'_1, \dots, e'_{|G|}$ be an enumeration of E that is increasing with w;
(3) for i from 1 to $|G| - 1$ do
    (a) choose $e_i \in E \setminus \{e_1, e_2 \dots, e_{i-1}\}$ such that $(V, \{e_1, \dots, e_i\})$ is acyclic and subject to this $w(e_i)$ is minimum
    (b) end do;
(4) output $T = (V, \{e_1, \dots, e_{|G|-1}\})$;

THEOREM 52. *The Minimal Spanning Tree algorithm computes a minimal spanning tree of the input graph G.*

PROOF. It suffices to prove by induction on i that there is a minimal spanning tree T^* such that $e_1, \dots, e_i \in E(T^*)$. The base step $i = 0$ is trivial, so suppose $i \geq 1$. If $e_i \in E(T')$ we are done, so suppose not. By induction there is a minimal spanning tree T' with $\{e_j : 1 \leq j \leq i - 1\} \subseteq E(T')$. By Proposition 50, there is $e' \in E(T') \setminus E(T)$ such that $T' - e' + e_i$ is a spanning tree. By hypothesis, $e_1, \dots, e_{i-1} \in T \cap T'$. So $(V, \{e_1, \dots, e_{i-1}, e'\})$ is acyclic. Thus e' was a candidate when we chose e_i for T. So $w(e_i) \leq w(e')$. Thus:

$$w(T') \leq w(T^*) := w(T') - w(e') + w(e_i) \leq w(T').$$

Thus $w(T^*) = w(W')$. So T^* is a minimal spanning tree with $e_1, \dots e_i \in E(T^*)$. □

PROPOSITION 53. *If T is a tree with k edges and G is a nontrivial graph with $\delta(G) \geq k$ then G contains a copy of T, i.e., a subgraph isomorphic to T.*

PROOF. Argue by induction on k.
Base Step: $k = 0$. Then $T \cong K_1$, so $T \cong G[\{v\}]$ for any vertex v.
Induction Step: $k > 1$. Let l be a leaf of T. Then $T' := T - l$ is a tree with $\|T'\| = k - 1$. By the induction hypothesis there exists $H' \subseteq G$ with $H' \cong T'$. Let p be the unique neighbor of l in T, and let x be the image of p in H'. Since $|H'| = \|H'\| + 1 = k$ and x is not adjacent to itself, x has at most $k - 1$ neighbors in H'. Since $\delta(G) \geq k$, there exists $y \in N_G(x) \setminus V(H')$. Set $H = H' + y + xy$. Then $H \subseteq G$ and we can extend the isomorphism between T' and H' to an isomorphism between T and H by mapping l to y. □

HW 20. (+) Let $d_1, \dots, d_n$ be positive integers with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \dots, d_n$ iff $\sum d_i = 2n - 2$.

HW 21. (*) Every tree is bipartite. Prove that every tree has a leaf in its larger partite set, and in both partite sets if they have equal size.

HW 22. (*) Let T be a tree such that $2k$ of its vertices have odd degree. Prove that T decomposes into k paths. Is it easier to prove this for forests?

HW 23. (+) Let T be a tree with even order. Prove that T has exactly one spanning subgraph such that every vertex has odd degree.

HW 24. (+) A *root* of a graph G is a special vertex r. A spanning tree T of a graph with root r is normal if every edge $xy \in E(G)$ satisfies either $x \in rTy$ or $y \in rTx$. Prove that every connected graph with root r has a normal spanning tree [Hint: Prove the stronger statement that every path P with end r is contained in a normal spanning tree.]

HW 25. (+) Let $\mathcal{T}$ be a set of subtrees of a tree G such that $T \cap T' \neq \emptyset$ for all $T, T' \in \mathcal{T}$. Prove that $\bigcap \mathcal{T} \neq \emptyset$.

HW 26. (*) A *directed graph* is a binary relation $G = (V, E)$. So the edges are *ordered* pairs. We still write xy for the directed edge (x, y). Let $E^+_G(x) := \{xy \in E\}$, $d^+_G(x) := |E^+_G(x)|$, and $\delta^+(G) := \min_{v \in V} d^+_G(v)\}$. A *directed cycle* $C = (V, E)$ is a digraph with the form $V = \{v_1, \dots, v_s\}$ and $E = \{v_i v_{i \oplus 1} : i \in [s]\}$. This directed cycle is also denoted by $v_1 \dots v_s v_1$ when it is clear form the context that the edges are directed.

Prove that a directed graph G has a directed cycle if $\delta^+(G) \geq 1$. [Hint: You might want to define the notion of a directed path.]

CHAPTER 3

Networks, Matching and Connectivity

In this Chapter we use the Max-Flow, Min-Cut Theorem to prove theorems about matchings and connectivity of graphs. First we introduce more notation for multigraphs and oriented edges.

Let $G = (V, E)$ be a multigraph. We call an edge with two distinct ends a *link* and an edge with two identical ends a *loop.* Since many edges may have the same ends, we need notation to denote the ends of an edge $e \in E$. Formally we write $e \in E(x, y)$ to mean that one end of e is x and the other end is y. This allows for the possibility that e is a loop and both its ends are x, i.e. $y = x$. Informally, we may write $e = xy$ to say that $e = xy$, but this is an abuse of the equality symbol.

It is natural to extend this notation to sets of vertices. For sets $A, B \subseteq V$ let

$$E(A, B) = \bigcup\{E(a, b) : (a, b) \in A \times B\}.$$

When one of the sets, say A, is a singleton $\{x\}$, we write $E(x, B)$. Thus our earlier notation has the form $E(x) = E(x, V)$. We set

$$\|A, B\|_G := \sum_{a\in A} |E(a, B)| = |E(A, B)| + \|G[A \cap B]\| = \sum_{b\in B} |E(A, b)|.$$

These definitions also apply to subsets $F \subseteq E$.

In this chapter every link $e \in E(x, y)$ will have two opposite *orientations*; loops have only one orientation. The set of all orientations of edges of E is denoted by $\vec{E}$. We refer to an arbitrary orientation of e by $\vec{e}$, and its opposite orientation by $\overleftarrow{e}$; if e is a loop then $\vec{e} = \overleftarrow{e}$. We may identify which orientation $\vec{e}$ is by writing $\vec{e} \in \vec{E}(x, y)$ if e is oriented from x to y; else $\overleftarrow{e} \in \vec{E}(x, y) = \overleftarrow{E}(y, x)$.

Let $\vec{F} \subseteq \vec{E}$ denote a subset of orientations. Then $\overleftarrow{F} = \{\overleftarrow{e} : \vec{e} \in \vec{F}\}$. Let $A, B \subseteq V$. Set $\overline{A} = V \smallsetminus A$ be the complement of A. Define

$$\vec{F}(A, B) = \{\vec{e} \in \vec{F} : \vec{e} \in \vec{E}(x, y) \text{ and } (x, y) \in A \times B \text{ and } a \neq b\},$$

abbreviate $\vec{F}(x, B) = \vec{F}(\{x\}, B)$, etc., and set $\vec{F}(x) = \vec{F}(x, \overline{\{x\}})$.

Consider a function $f : \vec{E} \to \mathbb{Z}$. Put

$$f(A, B) = \sum_{\vec{e}\in\vec{E}(A,B)} f(\vec{e}),$$

and as usual $f(x, B) = f(\{x\}, B)$, etc. Also $f(x) := f(x, V)$.

3.1. Max-Flow, Min-Cut Theorem

A *network* is a tuple $N = (V, E, s, t, c)$, where $G := (V, E)$ is the multigraph, $s \in V$ is the source, $t \in V$ is the sink, $s \neq t$, and $c : \vec{E} \to \mathbb{N}$ is the capacity function. A function $f : \vec{E} \to \mathbb{Z}$ is a *flow* in N if it satisfies:

(F1) $f(\vec{e}) = -f(\overleftarrow{e})$ for all links $\vec{e} \in \vec{E}$;
(F2) $f(v) = 0$ for all $v \in V \smallsetminus \{s, t\}$; and
(F3) $f(\vec{e}) \leq c(\vec{e})$ for all $\vec{e} \in \vec{E}$.

Notice that (F1) and (F3) imply $-c(\overleftarrow{e}) \leq f(\vec{e}) \leq c(\vec{e})$.

The *value of a flow f is* $v(f) = f(s)$. Let $\mathcal{F}(N)$ be the set of all flows in N. It is nonempty because the 0-flow is in $\mathcal{F}(N)$.

A *cut* in the network N is a pair $(S, \overline{S})$ with $s \in S \subseteq V$ and $t \in \overline{S}$. The *capacity of a cut* $(S, \overline{S})$ is $c(S, \overline{S})$. Let $\mathcal{S}(N)$ be the set of all cuts in N. It is nonempty as $(s, \overline{\{s\}}) \in \mathcal{S}$.

THEOREM 54. *Let $(S, \overline{S})$ be a cut in a network $N = (V, E, s, t, c)$ with flow f. Then*

$$v(f) = f(S, \overline{S}).$$

PROOF.

$$f(S, \overline{S}) = f(S, V) - f(S, S) =_{(F1)} f(s) + \sum_{v \in S - s} f(v) - 0 =_{(F2)} f(s) = v(f). \qquad \square$$

THEOREM 55 (Max-Flow, Min-Cut). *Every network $N = (V, E, x, y, c)$ satisfies*

$$\max_{f \in \mathcal{F}} v(f) = \min_{(S, \overline{S}) \in \mathcal{S}} c(S, \overline{S}).$$

PROOF. By Theorem 54, and the capacity constraint every flow f and cut $(S, \overline{S})$ satisfies

$$v(f) = f(S, \overline{S}) \leq_{(F3)} c(S, \overline{S}). \tag{3.1.1}$$

Now let f be a maximum flow in N. It suffices to construct a cut $(X, \overline{X})$ such that $v(f) = c(X, \overline{X})$. Let $\vec{D} = \{\vec{e} \in \vec{E} : f(\vec{e}) < c(\vec{e})\}$, and $\vec{H} = (V, \vec{D})$. Suppose $\vec{H}$ contains a directed s, t-path $\vec{W}$. Let $\varepsilon = \min\{c(\vec{e}) - f(\vec{e}) : \vec{e} \in \vec{E}(\vec{W})\}$. Obtain f' by increasing the flow on each edge of $\vec{W}$ by ε and decreasing the flow on each edge of $\overleftarrow{W}$ by $-\varepsilon$; as (F1–F3) are preserved, f' is a flow. Moreover, $v(f') = f'(s) = f(s) + \varepsilon > v(f)$, contradicting the maximality of f. So $\vec{H}$ contains no s, t-path.

Let $S = \{v \in V : \exists$ a directed s, v-path in $H\}$. Then $s \in S$ and $t \notin S$. So $(S, \overline{S})$ is a cut. Consider an oriented edge $\vec{e} \in (S, \overline{S})$; say $\vec{e} \in \vec{E}(x, y)$. As there is a directed s, x-path in $\vec{H}$, but no directed s, y-path in $\vec{H}$, $c(\vec{e}) = f(\vec{e})$. Thus $v(f) = f(S, \overline{S}) = c(S, \overline{S})$. $\square$

3.2. Matchings

A *matching* is a set of *links* with no common ends. A *maximal* matching is a matching that cannot be enlarged by adding an edge. A *maximum* matching is matching with maximum size among all matchings in the graph. A vertex is said to be *M-saturated* if and only if it is the end of an edge in M; otherwise it is *M-free*, and a set of vertices X is said to be *M-saturated* if every $x \in X$ is M-saturated. The matching M is *perfect* if every vertex is M-saturated.

Definition 56. Given a matching M in a graph G, an *M-alternating* path is a path P such that each vertex $v \in V(P)$ is incident to at most one edge in $E(P) \smallsetminus M$. Such a path is *M-augmenting* if its ends are M-free.

We will need the following easy proposition several times.

Proposition 57. *Every connected multigraph G with $\Delta(G) \leq 2$ is a path or cycle.*

Proof. Let $P = v_1 \dots v_t$ be a maximum path in G. As $\Delta(G) \leq 2$, no other edge is incident to the internal vertices of P. As P is maximum, no edge joins an end of P with a vertex of $V \smallsetminus V(P)$. So G is the path P or G is the cycle $P + v_1 v_t$. □

Theorem 58 (3.1.10 Berge). *A matching M in a graph G is not maximum iff G has an M-augmenting path.*

Proof. Suppose P is an M-augmenting path. Then

$$M' = M \triangle E(P) =_{def} (M \smallsetminus E(P)) \cup (E(P) \smallsetminus M)$$

is a larger matching.

Now suppose M is not maximum. Choose a matching M' with $|M'| > |M|$. Let H be the spanning submultigraph with edge set $M \cup M'$, where each $e \in M \cap M'$ is duplicated in H. Since each vertex is incident to at most one edge of each matching, $\Delta(H) \leq 2$. Using Proposition 57, the components of H are either alternating paths or cycles with the same number of edges from M as M'. Since $|M| < |M'|$, some component Q of H has more edges from M' than M. Such a component must be an M-augmenting path. □

HW 27. (*) Two players Alice and Bob play a game on a graph G. Alice begins the game by choosing any vertex. All other plays consist of the player, whose turn it is, choosing an unchosen vertex that is joined to the last chosen vertex. The winner is the last player to play legally. Prove that Alice has a winning strategy if G has no perfect matching, and Bob has a winning strategy if it does.

3.3. Bipartite matching

A bipartite G with bipartition $\{X, Y\}$ is called an X, Y-bigraph. For $S \subseteq X$ set $N(S) := \bigcup_{v \in S} N(v)$. For a function $f : A \to B$ and $S \subseteq A$, let $f(S) := \{y \in B : \exists x \in S(f(x) = y)\}$ be the range of f restricted to S.

Definition 59. A *cover* of a graph G is a subset $Q \subseteq V(G)$ that contains at least one end of every edge.

Let C be an odd cycle with $\|C\| = 2k + 1$. Since C is 2-regular, every set $Q \subseteq V(C)$ covers at most $2|Q|$ edges. Thus every vertex cover of C has at least $k + 1$ vertices. On the other hand, every matching M in C has $2|M|$ ends; so $|M| \leq k \leq k + 1 \leq |C|$.

Theorem 60 (König, Egerváry [1931] 3.1.16). *If G is a graph with a maximum matching M and a minimum cover W then $|M| \leq |W|$; if G is bipartite then $|M| = |W|$.*

Proof. Order V as $v_1 \prec \cdots \prec v_{|G|}$. Since W is a cover, every edge is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e)$ is the least $w \in e \cap W$. Since M is a matching, no vertex of W can be incident to two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose that G is an X,Y-bigraph. Define a network $N = (X \cup Y + s + t, E', s, t, c)$, where E' is the set of edges $E' = E(K(s,X)) \cup E_G(X,Y) \cup E(K(Y,t))$, and

$$c(\vec{e}) = \begin{cases} 1 & \text{if } \vec{e} \in \vec{E}'(s,X) \cup \vec{E}'(Y,t) \\ \infty := |X| + 1 & \text{if } \vec{e} \in \vec{E}'(X,Y) \\ 0 & \text{else} \end{cases}.$$

Let f be a maximum flow. By (F1) and (F3)

$$-0 \leq -f(\overleftarrow{e}) = f(\vec{e}) \leq 1 \text{ for } \vec{e} \in \vec{E}'(s,X) \cup \vec{E}'(Y,t) \text{ and}$$
$$-0 \leq -f(\overleftarrow{e}) = f(\vec{e}) \leq \infty \text{ for } \vec{e} \in \vec{E}'(X,Y).$$

By (F2), $f(x,Y) \in \{0,1\}$ for all $x \in X$ and $f(X,y) \in \{0,1\}$ for all $y \in Y$. It follows that $M' := \{xy \in E(X,Y) : f(x,y) = 1\}$ is a matching with $|M'| = f(X+s, Y+t) = v(f)$.

Let $(P, \overline{P})$ be a minimum cut. Now $(\{s\}, \overline{\{s\}})$ is a cut with $c(P,\overline{P}) \leq c(\{s\}, \overline{\{s\}}) = |X| < \infty$, and $c(\vec{e}) = \infty$ for all $\vec{e} \in \vec{E}'(X,Y)$, so $\vec{E}'(P,\overline{P}) \subseteq \vec{E}'(s, \overline{P} \cap X) \cup \vec{E}'(P \cap Y, t)$. Thus

$$N_G(P \cap X) \subseteq P \cap Y. \tag{3.3.1}$$

Thus $C := (P \cap Y) \cup (\overline{P} \cap X)$ is a cover: for any $xy \in E(X,Y)$ either $x \in \overline{P} \cap X \subseteq C$ or $x \in P$; in the latter case, $y \in N(x) \subseteq P \cap Y \subseteq C$ by (3.3.1). Using Theorem 55,

$$|W| \leq |P \cap Y| + |\overline{P} \cap X| = c(P,t) + c(s,\overline{P}) \leq c(P,\overline{P}) = v(f) = |M'| \leq |M| \leq |W|.$$

Thus equality holds throughout, and $|M| = |W|$. □

Corollary 61 (Hall). *Every X,Y-bigraph has a matching saturating X if and only if*

$$|S| \leq |N(S)| \text{ for all every set } S \subseteq X. \tag{3.3.2}$$

Proof. Let G be an X,Y-bigraph with a maximum matching M. If X is saturated then (3.3.2) holds since the vertices of any $S \subseteq X$ are matched to distinct elements of $N(S)$.

Otherwise, let W be a minimum cover. Then $\|X \smallsetminus W, Y \smallsetminus W\| = 0$. So $N(S) \subseteq Y \cap W$, where $S = X \smallsetminus W$. By Theorem 60, $|W| = |M| < |X|$. Thus

$$|X \cap W| + |X \smallsetminus W| = |X| > |W| = |X \cap W| + |Y \cap W|.$$

So (3.3.2) fails for S, since canceling $|X \cap W|$ yields

$$|S| > |Y \cap W| \geq |N(S)|.$$

□

A multigraph is k-regular if every vertex has degree k. A *k-factor of a graph* is a k-regular, spanning subgraph. So the edge set of a 1-factor is a perfect matching.

Corollary 62 (3.1.13). *Every k-regular bipartite multigraph G has a perfect matching.*

Proof. Suppose G is an k-regular X,Y-bimultigraph. Then

$$k|X| = \|X,Y\| = k|Y|.$$

It follows that $|X| = |Y|$. Thus it suffices to show that G has a matching that saturates X. By Hall's Theorem, it suffices to check (3.3.2). Consider any subset $S \subseteq X$. Then

$$k|S| = \|S,Y\| = \|S, N(S)\| \leq \|X, N(S)\| = k|N(S)|.$$

So $|S| \leq |N(S)|$. □

HW 28. (*) Let $\mathcal{S} = \{S_i : i \in I\}$ be a family of sets. A *system of distinct representatives* (sdr) for $\mathcal{S}$ is a sequence $(a_i : i \in I)$ of distinct elements such that $a_i \in S_i$ for all $i \in I$. Prove that if $\mathcal{S}$ is finite then $\mathcal{S}$ has an sdr if and only if $|J| \leq |\bigcup_{j\in J} S_j|$ for all $J \subseteq I$.

HW 29. (*) Prove that there is an injection $f : \binom{[2k+1]}{k+1} \to \binom{[2k+1]}{k}$ such that $f(S) \subseteq S$ for all $S \in \binom{[2k+1]}{k+1}$.

HW 30. (*) Let $\mathcal{P} = \{P_1, \dots, P_t\}$ and $\mathcal{Q} = \{Q_1, \dots, Q_t\}$ be two partitions of a set S into t subsets of size k. Prove that $\mathcal{P}$ and $\mathcal{Q}$ have a common sdr, that is there exist an sdr $(a_i : i \in [t])$ of $\mathcal{P}$ and a permutation $\sigma : [t] \to [t]$ such that $(a_{\sigma(i)} : i \in [t])$ is an sdr of $\mathcal{Q}$.

HW 31. (+) Suppose G is an X,Y-bigraph with $\delta(G) \geq 1$ such that every edge xy with $x \in X$, satisfies $d(x) \geq d(y)$. Prove that G has a matching that saturates every vertex in X. [Hint: Consider assigning each edge xy with $x \in X$ the weight $w(xy) = \frac{1}{d(x)}$.]

HW 32. (+) For $k, n \in \mathbb{N}$, let G be an A,B-bigraph with $|A| = n = |B|$ such that $\delta(G) \geq k$, and for all $X \subseteq A, Y \subseteq B$, if $|X|, |Y| \geq k$ then $|E(X,Y)| \neq \emptyset$. Prove: G has a perfect matching.

HW 33. (*) Let G be an X,Y-bigraph with $|X| = n = |Y|$. Prove that G a perfect matching if $\delta(G) \geq \frac{n}{2}$.

HW 34. (++) Prove that for all partitions $\mathcal{P} = \{P_i : i \in [n]\}$ and $\mathcal{Q} = \{Q_i : i \in [n]\}$ of a region S of area n into parts of area 1 there exists a permutation $\sigma : [n] \to [n]$ such that area $(P_i \cap Q_{\sigma(i)}) \geq f(n)$ for all $i \in [n]$, where

$$f(n) := \begin{cases} \frac{4}{(n+1)^2} & \text{if } n \text{ is odd} \\ \frac{4}{n(n+2)} & \text{if } n \text{ is even} \end{cases}.$$

Also show that the function f is optimal.

HW 35. (*) Prove that every bipartite graph has a matching of size $\frac{\|G\|}{\Delta(G)}$.

3.4. General matching

Notice that if H is a component of a graph G and $|H|$ is odd then G does not have a perfect matching.

Definition 63. Let $\mathcal{C}_G$ be the set of components of the graph G. A component with an odd number of vertices is said to be an *odd component*. Let $\mathcal{O}_G$ be the set of odd components of G and $o(G) = |\mathcal{O}_G|$.

Theorem 64 (Tutte [1947] 3.3.3). *A graph $G = (V, E)$ has a perfect matching iff*

$$o(G - S) \leq |S| \text{ for all } S \subseteq V. \tag{3.4.1}$$

Proof. If G has a perfect matching then (3.4.1) holds, since for any $S \subseteq V$, the odd components of $G - S$ must have vertices matched to distinct vertices in S. Now we assume (3.4.1), and show that G has a perfect matching by induction on $\|\overline{G}\|$.

Base Step: $G - U$ consists of disjoint complete subgraphs. By (3.4.1), $|U| \geq o(G - U)$; so one vertex in each odd component can be matched to a vertex in U. The remaining vertices of $G - U$ are in disjoint even cliques. By (3.4.1), $o(G \smallsetminus \emptyset) = |\emptyset| = 0$; so $|G|$ is even, and the remaining vertices of U form an even clique. Each of these even cliques has a perfect matching. Combining all these matchings yields a perfect matching of G.

Induction Step: $G-U$ has an incomplete component. Now G has a path aba' with $aa' \notin E$. As $b \notin U$, there is $b' \in V$ with $bb' \notin E$. Adding aa' or bb' does not increase $o(G-S)$ for any $S \subseteq G$, so by induction $G+aa'$ and $G+bb'$ have perfect matchings $M^+ = M + aa'$ and $L^+ = L + bb'$, respectively. In G, the M-free vertices are a and a', and the L-free vertices are b and b'. The component of $G[M \cup L]$ containing a is an alternating path $P = a \dots x$ with $x \in \{a', b, b'\}$. If $x \in \{a', b\}$ then aPx or $aPxa'$ is M-augmenting; else $baPb'$ is L-augmenting. Anyway, G has a perfect matching. □

The next theorem is a generalization of Tutte's Theoerm. We will give two proofs. The first builds on the previous proof. The second starts from scratch.

THEOREM 65 (Tutte [1947] 3.3.3+3.3.7). *Let $G = (V, E)$ be a graph with a maximum matching M. Then the number of M-free vertices of G is equal to*

$$d := \max_{S \subseteq V} o(G-S) - |S|. \tag{3.4.2}$$

FIRST PROOF. If $d = 0$ then we are done by Theorem 64. So assume $d > 0$. Choose $S \subseteq V$ so that $d = o(G-S) - |S|$. Every matching has at least d free vertices, since each odd component has a vertex that is free or is matched to a vertex of S.

Let $G^+ := G \vee K_d(Q)$, and consider $T \subseteq V(G^+)$. As $d \geq 1$, G^+ is connected. Also

$$|G^+| = |G| + d \equiv o(G-S) + |S| + d = 2d + 2|S| \equiv 0 \mod 2.$$

Thus if $T = \emptyset$ then $o(G-T) - |T| = 0$. If $o(G^+ - T) \geq 2$, then $Q \subseteq T$, and so

$$o(G^+ - T) - |T| \leq (o(G - (T \setminus Q)) - |T \setminus Q|) - |Q| \leq d - d \leq 0.$$

Otherwise, $o(G^+ - T) - |T| \leq 1 - 1 = 0$. By Theorem 64, G^+ has a perfect matching M^+. Then $M := M^+ \setminus E(Q, V)$ is a matching in G with at most $|Q| = d$ unsaturated vertices. □

DEFINITION 66. A graph G is *factor critical* if $G - v$ has a perfect matching for every vertex $v \in V(G)$. A set S is *matchable* into $\mathcal{O}_{G-S}$ if there exists a matching M saturating S such that each edge $e \in M$ has one end in S and one end in an odd component, and at most one vertex of each odd component is saturated.

SECOND PROOF. For any set $S \subseteq V$ and matching M, there are at least $o(G-S) - |S|$ unsaturated vertices: each odd component $H \subseteq G - S$ has an M-unsaturated vertex, unless $M \cap E(S, V(H)) \neq \emptyset$, and there are at most $|S|$ such edges in M. So it suffices to show that there exists a set $S \subseteq V$ and a matching M with exactly $o(G-S) - |S|$ unsaturated vertices.

Argue by induction on $|G|$. For the base step $|G| = 1$, let $S = \emptyset$. Then $o(G-S) - |S| = 1$ and the only vertex of G is unsaturated by any matching. Now consider the induction step.

Choose a set $S \subseteq V$ so that $o(G-S) - |S|$ is maximum, and subject to this, $|S|$ is also maximum. We first prove the following three claims:

CLAIM (1). Every component of $G - S$ is odd.

PROOF. Suppose $H \in \mathcal{C}_{G-S}$ with $|H|$ even. Choose a non-cut vertex x (end of a maximal path) of H, and set $S' = S + x$. Then

$$\mathcal{O}_{G-S'} = \mathcal{O}_{G-S} + (H - x) \text{ and } |S'| = |S| + 1.$$

Thus $o(G-S) - |S| = o(G-S') - |S'|$, contradicting the choice of S, since $|S| < |S'|$. □

Claim (2). Every odd component of $G-S$ is factor critical.

Proof. Consider any $H \in \mathcal{O}_{G-S}$ and any vertex $x \in V(H)$. We must show that $H' = H - x$ has a perfect matching. By the induction hypothesis, it suffices to show that $o(H'-T) - |T| \le 0$ for all $T \subseteq V(H')$. So consider any such T, and set $S' = S \cup T + x$. Then $|S'| = |S| + |T| + 1 > |S|$, and so by the choice of S

$$o(G-S) - |S| > o(G-S') - |S'|.$$

Since $T + x \subseteq V(H)$,

$$\mathcal{O}_{G-S'} = (\mathcal{O}_{G-S} - H) \cup \mathcal{O}_{H'-T}.$$

So

$$\begin{gathered} o(G-S) - |S| > o(G-S') - |S'| = o(G-S) - 1 + o(H'-T) - |S| - |T| - 1 \\ 2 > o(H'-T) - |T|. \end{gathered}$$

Moreover, by Claim (1), H is an odd component, and so $|H'|$ is even. Thus

$$o(H'-T) - |T| \equiv |H'-T| + |T| \equiv |H'| \equiv 0 \mod 2.$$

Hence $1 \neq o(H'-T) - |T|$, and so $0 \ge o(H'-T) - |T|$. □

Claim (3). S is matchable into $\mathcal{O}_{G-S}$.

Proof. Let H be the $S, \mathcal{O}_{G-S}$-bigraph with edge set

$$F := \{xD : x \in S, D \in \mathcal{O}_{G-S} \text{ and } N(x) \cap V(D) \neq \emptyset\}.$$

We will show H has a matching saturating S by checking (3.3.2) of Corollary 61. Consider any $T \subseteq S$, and set $S' = S - T$. Then $\mathcal{O}_{G-S} \setminus N_H(T) \subseteq \mathcal{O}_{G-S'}$. By the choice of S

$$\begin{gathered} o(G-S) - |S| \ge o(G-S') - |S'| \ge o(G-S) - N_H(T) - |S| + |T| \\ |N_H(T)| \ge |T|. \end{gathered}$$

□

Finally, we obtain a matching M as follows. By Claim (3) there is a matching M_0 that saturates S and one vertex of $|S|$ odd components. For each $H \in \mathcal{O}_{G-S}$ choose a vertex v_H, and if possible, choose v_H so that it is M_0-saturated. Next use Claim (2) to obtain matchings M_H of $H - v_H$ for every odd component $H \in \mathcal{O}_{G-S}$. Then

$$M := M_0 \cup \bigcup_{H \in \mathcal{O}_{G-S}} M_H$$

is matching of G. Using Claim (1), it saturates every vertex of G except those $o(G-S)-|S|$ vertices v_H that are not saturated by M_0. □

We have actually proved a stronger statement.

Theorem 67. *Let $G = (V, E)$ be a graph and $S^* \subseteq V$ be a set of vertices such that for all $S \subseteq V$*

(1) $o(G-S) - |S| \le o(G-S^*) - |S^*|$; and
(2) if equality holds in (1) then $|S| \le |S^*|$.

Then every component of $G - S^*$ is factor-critical; and every maximum matching saturates S^*, matches S^* into $\mathcal{O}_{G-S^*}$, and leaves at most one vertex of each component of $G - S^*$ free.

A graph $G = (V, E)$ is transitive if for all $x, y \in V$ there is an automorphism φ of G with $\varphi(x) = y$. For example C_n, $K_{n,n}$ and the Petersen graph, but not P_n, are transitive.

HW 36. (*) Prove that a transitive graph does not have a cut vertex.

HW 37. (+) Prove that if G is a connected, transitive graph with $|G|$ even then G has a perfect matching. (Lovasz has conjectured that every connected transitive graph has a hamiltonian path; there are only four known examples of such graphs that are not hamiltonian.) [Hint: Use Theorem 65 to show that if G does not have a perfect matching then some, but not all, of its vertices have the property that they are saturated in every maximum matching.]

3.5. Applications of Matching Theorems

A cut-edge is also called a *bridge*. A *bridgeless* graph is a graph without cut-edges. It need not be connected.

THEOREM 68 (Petersen [1891] 3.3.8). *Every bridgeless cubic graph $G = (V, E)$ contains a 1-factor.*

PROOF. By Tutte's Theorem, it suffices to show that $o(G - S) \leq |S|$ for every subset $S \subseteq V$. Fix any such S and consider any $H \in \mathcal{O}_{G-S}$. Since G is cubic and $|H|$ is odd,

$$3|H| = \|H, V\| = 2\|H\| + \|H, S\| \equiv 1 \mod 2.$$

It follows that $\|H, S\|$ is odd, and since G is bridgeless, $\|H, S\| \geq 3$. Thus

$$3o(G - S) \leq \|S, V \smallsetminus S\| \leq 3|S|,$$

and so $o(G - S) \leq |S|$. □

THEOREM 69 (Petersen [1891] 3.3.9). *Every regular multigraph with positive even degree has a 2-factor.*

PROOF. Suppose G' is $2k$-regular with $k \in \mathbb{Z}^+$. It suffices to show that each component $G = (V, E)$ of G' has a 2-factor. By Euler's Theorem 36, G has an Eulerian trail $T = v_1 \dots v_n (= v_1)$. Let $V' = \{v' : v \in V\}$ and $V'' = \{v'' : v \in V\}$ be sets of new vertices, disjoint from V and each other, where $v \mapsto v'$ and $v \mapsto v''$ are injections. Define a V', V''-bigraph by $E(H) = \{e_i^* : i \in [n]\}$, where $e_i^* \in E_H(v_i', v_{i\oplus 1}'')$. Since each vertex v of G is incident to $2k$ edges, it appears k times in T. Say $v = v_{i_1} = \cdots = v_{i_k}$. Then

$$E(v) = \{v_{i_1-1}v_{i_1}, v_{i_1}v_{i_1+1}, \dots, v_{i_k-1}v_{i_k}, v_{i_k}v_{i_k+1}\}$$

and

$$E_H(v') = \{v'v''_{i_1+1}, \dots, v'v''_{i_k+1}\} \text{ and } E_H(v'') = \{v''v'_{i_1-1}, \dots, v''v'_{i_k-1}\}.$$

So H is k-regular. By the Corollary 62, H has a perfect matching M. Let $F = \{xy \in E : x'y'' \in M\}$. Then (V, F) is a 2-factor of G: for each $y \in V$ there exists a unique x such that $x'y'' \in M$ and a unique z such that $y'z'' \in M$. Since T is a trail, and $xy, yz \in T$, we have $xy \neq yz$ (maybe xy and yz were parallel edges if G is a multi graph). □

HW 38. (*) Prove that a 3-regular graph has a 1-factor iff it decomposes into copies of P_4.

HW 39. (+) Suppose G is a graph on $2k$ vertices with $k \geq 3$, whose complement $\overline{G}$ does not have a 1-factor. Let S be the set whose existence is guaranteed by Tutte's Theorem (applied to $\overline{G}$). Prove that

(1) If $|S| = 0$ then G contains $K_{c,2k-c}$ for some odd c.
(2) If $|S| \geq k - 1$ then G contains K_{k+1}.
(3) If $1 \leq |S| \leq k-2$ then $\Delta(G) \geq k+1$. [Hint: $\frac{x+1}{x+2}(2k-x) > k$ when $1 \leq x \leq k-2$.]

HW 40. (+) Let M be a matching in a graph G with an M-unsaturated vertex u. Prove that if G has no M-augmenting path starting at u then G has a maximum matching L such that u is L-unsaturated.

HW 41. (+) A graph is *claw-free* if it does not contain an induced $K_{1,3}$. Prove that a connected claw-free graph of even order has a 1-factor. Find (easy) a small counter example if the graph is not connected.

HW 42. (+) Let G be a k-regular graph with $|G|$ even that remains connected when any $k-2$ edges are removed. Prove that G has a 1-factor.

3.6. Introduction to Connectivity

Definition 70. A *separating set* or *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* $\kappa(G)$ is the minimum size of a vertex set S such that $G - S$ has more than one component or only one vertex. A graph G is k-connected if $k \leq \kappa(G)$. Two vertices x and y are *separated* by S if they are in different components of $G - S$.

Note that it is not possible to disconnect a complete graph by removing vertices. The definition sets the connectivity of a complete graph K equal to $|K| - 1$.

Definition 71. A *disconnecting set* of edges in a graph G is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. The *edge-connectivity* $\kappa'(G)$ of G is the minimum size of a disconnecting set of edges. It is k-edge-connected if $k \leq \kappa'(G)$. Two vertices x and y are *separated* by F if they are in different components of $G - F$.

An *edge cut* in G is a set of edges of the form $E(S, \overline{S})$, where $\emptyset \neq S \neq V(G)$.

Theorem 72 (Whitney [1932] 4.1.9). *Every graph $G = (V, E)$ satisfies*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. Choose a vertex $v \in V$ with $d(v) = \delta(G)$. Then $E(v)$ is a disconnecting set of edges of size $\delta(G)$, and so $\kappa'(G) \leq \delta(G)$.

For the first inequality, consider a minimum edge cut $E(S, \overline{S})$; so $\left\|S, \overline{S}\right\| = \kappa'(G)$. Note that $\kappa(G) \leq |G| - 1$. First suppose that every vertex in S is adjacent to every vertex in $\overline{S}$. Using calculus,

$$\kappa'(G) = \left\|S, \overline{S}\right\| = |S||\overline{S}| \geq |G| - 1 \geq \kappa(G).$$

Else there are $x \in S$ and $y \in \overline{S}$ with $xy \notin E$. Define $f : E(S, \overline{S}) \to V$ by $f(e) = z$ if $e = xz$; else $f(e) \in e \cap S$. So $f(e) \in e \smallsetminus \{x, y\}$. Every x, y-path P contains an edge $e \in E(S, \overline{S})$, and $f(e)$ is an interior vertex of P. So range(f) separates x from y. Thus

$$\kappa'(G) = \left\|S, \overline{S}\right\| \geq |\text{range}(f)| \geq \kappa(G).$$ □

THEOREM 73. *Every cubic graph $G = (V, E)$ satisfies $\kappa(G) = \kappa'(G)$.*

PROOF. Put $\kappa := \kappa(G)$ and $\kappa' := \kappa'(G)$. If $\kappa = 0$ then G is disconnected, and so $\kappa' = 0$ also. If $\kappa = 3$ then $3 = \kappa \leq \kappa' \leq \delta(G) = 3$, and again $\kappa = \kappa'$. So assume $\kappa \in [2]$. Let S be a separating set with $|S| = \kappa$, and let H_1, H_2 be two components of $G - S$. Since S is minimum, every vertex $v \in S$ has a neighbor in each H_i. Since G is cubic, there exists i such that v has a unique neighbor w_v in H_i. Choose such a w_v, preferring $w_v \in H_1$, and set $F := \{vw_v : v \in S\}$. Then $|F| = |S|$. Moreover F is a disconnecting set of edges: for each $i \in [2]$, F disconnects H_i from S unless $w_{v_i} \notin H_i$ for some $v_i \in S$. In this case, $S = \{v_1, v_2\}$, and two neighbors in H_1 by the preference for H_1. Since $\delta(G) = 3$ and $v_1 w_{v_1} \in H_2$, $v_1 v_2 \notin E$. Thus F separates v_1 from v_2. So $\kappa' \leq |F| = |S| = \kappa \leq \kappa'$. □

LEMMA 74 (Expansion Lemma 4.2.3). *If G is k-connected and G' is obtained from G by adding a new vertex x with at least k neighbors in G then G' is k-connected.*

PROOF. Since $|G'| = |G| + 1$, it suffices to show that G' does not have a $(k-1)$-separating set. Consider any $(k-1)$-set S. Then $G - S$ is connected and x has a neighbor in $G - S$, so $G' - S$ is connected. □

HW 43. Prove that an r-connected graph on an even number of vertices with no induced subgraph isomorphic to $K_{1,r+1}$ has a 1-factor.

3.7. Low Connectivity

For graphs $H \subseteq G$, a path $v_1 \dots v_n \subseteq G$ is an H-path if $v_1, v_n \in H$ and $v_2 \dots v_{n-1} \subseteq G - H$.

DEFINITION 75. A sequence of graphs $G_0, \dots, G_t$ is a 2-witness for a graph G iff

(1) G_0 is a cycle and $G_t = G$; and
(2) for all $i \in [t]$ there is a G_{i-1}-path P_i with $G_i = G_{i-1} + P_i$.

THEOREM 76 (Whitney [1932] 4.2.8). *A graph $G = (V, E)$ is 2-connected iff it has a 2-witness set.*

PROOF. First suppose G is 2-connected. Then $\delta(G) \geq 2$, and so G contains a cycle C. Let $H \subseteq G$ be a maximal subgraph such that H has a 2-witness $G_0, \dots, G_t$. It exists because C is a candidate. It suffices to show that $H = G$.

Suppose $v_0 \in V(G - H)$. Since G is connected, there exists a v_0, H-path $Q = v_0 \dots v_s$. Since G is 2-connected, $G - v_s$ is connected. So there exists a v_{s-1}, H-path P in $G - v_s$. Then $P_{t+1} := v_s v_{s-1} P$ is an H-path in $G_{t+1} := H + P_{t+1}$, contradicting the maximality of H. We conclude that H is a spanning subgraph of G.

Now suppose $xy \in E(G - H)$. Then xy is an H-path of $G_{t+1} := H + xy$, contradicting the maximality of H. So H is an induced, spanning subgraph of G. Thus $G = H$.

Now suppose G has a 2-witness $G_0, \dots, G_t$. Argue by induction on t that G is 2-connected. The base step $t = 0$ is easy since the cycle G_0 is 2-connected. So consider the induction step $t \geq 1$. By the induction hypothesis, G_{t-1} is 2-connected. Say $G = G_{t-1} + P_t$, where $P_t := v_1 \dots v_s$. Consider any $x \in V(G)$. We must show that $G - x$ is connected. Since G_{t-1} is 2-connected, $G_{t-1} - x$ is connected. Also every vertex in $P_t - x$ is connected to a vertex $v \in \{v_1, v_s\} \subseteq G_{t-1} - x$ in G, even if $x \in V(P_t)$. It follows that G is 2-connected. □

Definition 77. Let $e = xy$ be an edge in a graph G, and fix a new vertex v_e. The graph $G \cdot e$ obtained by *contracting* e is defined by

$$G \cdot e := (G - x - y) \cup K(v_e, N_G(\{x, y\}) - x - y).$$

Note that if P' is a path in $G \cdot e$ then either P' is a path in G or $v_e \in V(P')$. In the latter case we can obtain a path in G by replacing v_e by one of x, y, xy, yx. If P is a path in G then either P is a path in $G \cdot e$ or one or both of x, y are in $V(P)$. In the latter case we can obtain a path P' in $G \cdot e$ by replacing one of x, y, xPy, yPx by v_e.

Lemma 78. *Let $G = (V, E)$ be a graph and $xy \in E$. If S' is a separating set of $G' := G \cdot xy$ then either $S' \subseteq V$ and S' is a separating set of G or $v_{xy} \in S'$ and $S := S' - v_{xy} + x + y$ is a separating set of G.*

Proof. Suppose $v_{xy} \notin S'$. Then $S' \subseteq V(G)$ and v_{xy} is in a component H of $G' - S'$. Choose a vertex w in another component of $G' - S'$; then $w \in G$. Consider any $w, \{x, y\}$-path $P = w \dots u'u$ in G. Then $P' := wPu'v_{xy}$ is a w, v_{xy} path in G'. Since $S' \subseteq V$ is a separating set in G', there is $v \in S' \cap V(P) \subseteq V$. Thus S' separates x from w in G.

Else $v_{xy} \in S$. Let u and v be vertices in distinct components of $G' - S'$. If P is a u, v-path in G then there is a u, v-path P' in G' such that $V(P') \subseteq V(P) + v_{xy}$, and $v_{xy} \in V(P')$ only if x or y is in $V(P)$. As S' separates u and v in G', there is $w \in V(P') \cap S'$. If $w \neq v_{xy}$ then $w \in S$. If $w = v_{xy}$ then x or y is in $V(P) \cap S$. So S separates u and v in G. □

Lemma 79 (Thomassen [1980] 6.2.9). *Every 3-connected graph G with $|G| \geq 5$ has an edge e such that $G \cdot e$ is 3-connected.*

Proof. Suppose not. Consider any edge xy. Since $G' := G \cdot xy$ is not 3-connected and $|G'| \geq 4$, G' has a separating 2-set S'. By Lemma 78, S' has the form $S' = \{v_{xy}, z\}$ and $S := \{x, y, z\}$ is a separating 3-set in G.

So far the edge $xy \in E$ and the 2-separating set S' are arbitrary. Now choose xy and $S' = \{v_{xy}, z\}$, and set $S = \{x, y, z\}$ as above, so that $G - S$ has a component H that is as large as possible among all possible choices of xy and S'. Let H' be another component of $G - S$. Since S is a minimal separating set, each of x, y, z has a neighbor in each of H and H'. Let u be a neighbor of z in H'. Then $G \cdot uz$ has a separating set $\{v_{uz}, v\}$, and $\{u, v, z\}$ is a separating set for G.

Put $H^+ = G[H + x + y]$. Then H^+ is connected, and $u, z \notin V(H^+)$. Thus $H^* := H^+ - v$ is disconnected, since otherwise H^* is contained in a component C of $G - \{u, v, z\}$ with $|C| \geq |H| + |\{x, y\} - v| \geq |H| + 1$, contradicting the choice of xy, z, H. As x and y are adjacent, they are in the same component of H^*; let U be a different component of H^*. Suppose $ab \in E(U, V \smallsetminus U)$ with $a \in U \subseteq H$. Then $b \in V(H) \cup S = V(H^+) + z$. If $b \in H^+$ then $b = v$; else $b = z$. Thus $\{v, z\}$ separates U from H', contradicting $\kappa(G) \geq 3$. □

Definition 80. A sequence of graphs $G_0, \dots, G_s$ is a 3-*witness for a graph G* iff

(1) $G_0 = K_4$ and $G_s = G$; and
(2) for all $i \in [s]$ there is $xy \in E(G_i)$ with $G_{i-1} = G \cdot xy$ and $d_{G_i}(x), d_{G_i}(y) \geq 3$.

Theorem 81. *A graph G is 3-connected iff it has a 3-witness.*

Proof. First suppose that G is 3-connected. Then $|G| \geq 4$. We show by induction on $|G|$ that G has a 3-witness. Suppose $|G| = 4$. If $xy \notin E(G)$ then $V(G) \smallsetminus \{x, y\}$ is a 2-set that separates x from y, a contradiction. So $G = K_4$, and $G_0 = K_4 = G$ is a 3-witness for G. Otherwise, $|G| \geq 5$. By Lemma 79, there exists an edge $xy \in E(G)$ such that $G \cdot xy$ is 3-connected. Since G is 3-connected, $d(x), d(y) \geq 3$. By induction, $G \cdot xy$ has a 3-witness $G_0, \dots, G_s$. So $G_0, \dots, G_s, G$ is a 3-witness for G.

Now let $G_0, \dots, G_s$ be a 3-witness for G. Argue by induction on s that G is 3-connected. If $s = 0$ then $K_4 = G_0 = G$ is 3-connected. Otherwise, for some edge $xy \in E(G)$, both $G_{s-1} = G \cdot xy$ and $d_G(x), d_G(y) \geq 3$. By induction $G \cdot xy$ is 3-connected. Suppose for a contradiction that S is a 2-separator in G. If $S = \{x, y\}$ then v_{xy} is a cut vertex of $G \cdot xy$, a contradiction. So there is a component H of $G - S$ containing at least one, say x, of x and y, and another component H' containing neither x nor y. As $d_G(x) \geq 3$ and $N(x) \subseteq V(H) \cup S$, x has a neighbor v in H. If $y \in H$ then S separates v_{xy} from H' in $G \cdot xy$; else $y \in S$, and $S' := S - y + v_{xy}$ separates v from H' in $G \cdot xy$. Anyway, $\kappa(G \cdot xy) \leq 2$, a contradiction. □

The last paragraph of the above proof is subtle. If $d_G(x) < 3$ then we could have $S = N(x)$, and $V(H) = \{x\}$. Then $H - x$ is not a component of $G \cdot xy - S'$ since $H - x$ has no vertices.

Conjecture 82 (Lovasz). *There exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all $k \in \mathbb{Z}^+$ and $f(k)$-connected graphs G and all vertices $x, y \in V(G)$, there exists a partition $\{V_1, V_2\}$ of $V(G)$ such that $G[V_1]$ is an x, y-path and $G[V_2]$ is k-connected.*

HW 44. Show that Conjecture 82 is true in the case $k = 1$ with $f(1) := 3$.

3.8. Menger's Theorems

Definition 83. A *directed x, y-path* is a directed graph $\vec{P} = (\{v_1 \dots v_t\}, \{\vec{e}_1, \dots, \vec{e}_{t-1}\})$, where $\vec{e}_i \in E(v_i, v_{i+1})$.

Definition 84. Two x, y-paths (directed or not) are edge disjoint if they have no common edges. (They may have common vertices.) They are *independent* or *internally disjoint* if they have no common internal vertices. Define the following related parameters:

(1) $\vec{\lambda}'(x, y)$ is the maximum size of a set of edge-disjoint, directed x, y-paths;
(2) $\lambda'(x, y)$ is the maximum size of a set of edge-disjoint x, y-paths;
(3) $\lambda(x, y)$ is the maximum size of a set of independent x, y-paths;
(4) $\vec{\lambda}(x, y)$ is the maximum size of a set of independent, directed x, y-paths.

Definition 85. Consider a graph $G = (V, E)$ and an orientation $\vec{G}$ of G. Let $x, y \in V$ be distinct. A set $\vec{F} \subseteq \vec{E}$ *is a directed x, y-disconnecting set* if there are no directed x, y-paths in $\vec{G} - \vec{F}$; a set $F \subseteq E$ is an *x, y-disconnecting set* if there are no x, y-paths in $G - F$. Now suppose $xy \notin E$. A subset $S \subseteq V(G) - \{x, y\}$ is an *x, y-separating set* if there is no x, y-path in $G - S$. Now assume $\vec{xy} \notin \vec{E}$. A subset $S \subseteq V(G) - \{x, y\}$ is is a *directed x, y-separating set* if there is no directed x, y-path in $G - S$. Define the following related parameters:

(1) $\vec{\kappa}'(x, y)$ is the minimum size of a directed x, y-disconnecting set;
(2) $\kappa'(x, y)$ is the minimum size of an x, y-disconnecting set;
(3) if $xy \notin E$ then $\kappa(x, y)$ is the minimum size of an x, y- separating set;
(4) if $\vec{xy} \notin E$ then $\vec{\kappa}(x, y)$ is the minimum size of a directed x, y-separating set.

If $ab \in E(G)$ then no $S \subseteq V$ separates a and b. This is why $\kappa(a, b)$ is only defined when $ab \notin E$.

Suppose $\vec{P} = (V, \vec{E})$ is a directed graph. Then $P = (V, \{e : \vec{e} \in E\})$. For a (undirected) graph $G = (V, E)$ let $\overleftrightarrow{G} = (V, \{\vec{e}, \overleftarrow{e} : e \in E\})$. Notice that if $\vec{G}$ is a directed graph then it is possible that $\overleftrightarrow{G} \neq \vec{G}$; for instance $\vec{G}$ could be a directed path. For an x, y-path P let $\overrightarrow{P}$ be the directed x, y-path obtained by directing the edges of P toward y. Now, if $\vec{P}$ is a directed x, y-path then $\overrightarrow{P} = \vec{P}$.

THEOREM 86 (Menger 1927, directed edge). *Every directed graph $G(V, \vec{E})$ satisfies $\vec{\lambda}'(x, y) = \vec{\kappa}'(x, y)$ for all distinct $x, y \in V$.*

PROOF. Consider the network $N = (G, x, y, c)$, where $c(\vec{e}) = 1$ for all $\vec{e} \in \vec{E}$. For a flow f let $H(f)$ be the spanning graph of G whose edge set is $\vec{E}(H(f)) = \{\vec{e} \in \vec{E} : f(\vec{e}) = 1\}$. Among all maximum flows choose a flow f such that $\|H(f)\|$ is minimal. Then $H(f)$ has no directed cycle: Suppose $\vec{C}$ is a directed cycle in $H(f)$. Each $\vec{e} \in E(\vec{C})$ satisfies $f(\vec{e}) = 1 = -f(\overleftarrow{e})$. Set

$$f'(\vec{e}) = \begin{cases} f(\vec{e}) & \text{if } \vec{e} \notin \vec{E}(\overleftrightarrow{C}) \\ 0 & \text{if } e \in E(\overleftrightarrow{C}) \end{cases}.$$

Then f' is a flow (check (F1–F3)) and $v(f') = v(f)$, but $\|H(f')\| = \|H(f)\| - \|\vec{C}\|$, contradicting the choice of f.

Let $(U, \overline{U})$ be a minimum cut in N. By the Max-flow, Min-Cut Theorem, $c(U, U') = v(f)$. Also every directed x, y-path contains an edge of $\vec{F} := \vec{E}(U, \overline{U})$. Thus $\vec{F}$ is a directed x, y-disconnecting set. So $\vec{\kappa}'(x, y) \leq c(U, \overline{U})$.

Let $\mathcal{P}$ be a set of $\vec{\lambda}'(x, y)$ edge-disjoint directed x, y-paths and $\vec{S}$ be a directed x, y-disconnecting of size $\vec{\kappa}'(x, y)$. Then every path $P \in \mathcal{P}$ contains an edge of $\vec{S}$. So $\vec{\lambda}'(x, y) \leq \vec{\kappa}'(x, y)$.

Now $\vec{\lambda}'(x, y) \leq \vec{\kappa}'(x, y) \leq c(U, \overline{U}) = v(f)$. Thus it suffices to show that $v(f) \leq \vec{\lambda}'(x, y)$. Argue by induction on $l := \vec{\lambda}'(x, y)$. For the base step $l = 0$, arguing by contraposition, suppose $v(f) \geq 1$. For each $v \in G$ with $d^+_{H(f)}(v) \geq 1$ fix one edge $\vec{e}_v \in \vec{E}(v, V - v)$. By (F2), if $\vec{e}_v = v\vec{w}$ then $\vec{e}_w$ exists unless $w = y$. Consider the maximum directed walk $\vec{W} = v_1 \dots v_t$, where $v_1 = x$ and $v_i\vec{v}_{i+1} = \vec{e}_{v_i}$ for all $i \in [t-1]$. As $v(f) \geq 1$, it has at least one edge $x\vec{v}_2$. Since $H(f)$ does not contain a directed cycle $\vec{W}$ is a directed path. Since G is finite, $\vec{W}$ must end. Since it does not end before y, it is a directed x, y-path. So $\vec{\lambda}'(x, y) \geq 1$.

Now suppose $l \geq 1$ and let $P = xx_1 \dots y \in \mathcal{P}$. Set $G' = G - \vec{E}(P)$ and $N' = (G', x, y, c)$. Then $v_N(f) = v_{N'}(f) + f(x\vec{x}_1) = v_{N'}(f) + 1$ since $\vec{E}_N(x, V - x) = \vec{E}_{N'}(x, V - x) + x\vec{x}_1$. If $\mathcal{P}'$ is a set of directed, edge-disjoint x, y-paths in G' then $\mathcal{P}' + P$ is a set of directed, edge-disjoint x, y-paths in G. Thus $\vec{\lambda}'_{G'}(x, y) + 1 \leq \vec{\lambda}'_G(x, y)$. Let f' be a maximum flow in N'. Then $v_{N'}(f) \leq v_{N'}(f')$. Now, using induction,

$$v_N(f) = v_{N'}(f) + 1 \leq v_{N'}(f') + 1 =_{i.h.} \vec{\lambda}'_{G'}(x, y) + 1 \leq \vec{\lambda}'_G(x, y). \qquad \square$$

THEOREM 87 (Menger 1927, edge). *Every graph $G(V, E)$ satisfies $\lambda'(x, y) = \kappa'(x, y)$ for all distinct $x, y \in V$.*

PROOF. It suffices to prove

$$\vec{\kappa}_{\overleftrightarrow{G}}(x, y) =_{(0)} \vec{\lambda}_{\overleftrightarrow{G}}(x, y) \leq_{(1)} \lambda'_G(x, y) \leq_{(2)} \kappa'_G(x, y) \leq_{(3)} \vec{\kappa}'_{\overleftrightarrow{G}}(x, y).$$

Theorem 86 implies (0). As every x,y-path meets every x,y-disconnecting set, (2) holds.

For (1), let $\vec{\mathcal{P}}$ be a set of $\vec{\lambda}'_{\overleftrightarrow{G}}(x,y)$ edge-disjoint, directed x,y-paths in $\overleftrightarrow{G}$ with $\|\bigcup\{\vec{E}(\vec{P}) : \vec{P} \in \vec{\mathcal{P}}\|$ minimum, and set $\mathcal{P} = \{P : \vec{P} \in \vec{\mathcal{P}}\}$. Suppose some $P, Q \in \mathcal{P}$ have a common edge vw. Since $\overrightarrow{P}$ and $\overrightarrow{Q}$ are edge-disjoint, (say) $v\vec{w} \in \vec{E}(\overrightarrow{P})$ and $w\vec{v} \in \vec{E}(\overrightarrow{Q})$. Then $\overrightarrow{\mathcal{P}} + x\overrightarrow{P}v\overrightarrow{Q}y + x\overrightarrow{Q}w\overrightarrow{P}y - \overrightarrow{P} - \overrightarrow{Q}$ is a set of $\vec{\lambda}_{\overleftrightarrow{G}}(x,y)$ edge-disjoint, directed x,y-paths in $\overleftrightarrow{G}$ with two less edges, contradicting the choice of $\vec{\mathcal{P}}$. So $\vec{\lambda}'_{\overleftrightarrow{G}}(x,y) = |\mathcal{P}| \le \lambda'_G(x,y)$.

For (3), let $\vec{F}$ be a directed x,y-disconnecting set of size $\vec{\kappa}'(x,y)$, and consider any x,y-path P in G. Then there is an edge $v\vec{w}$ in $\vec{E}(\vec{P}) \cap \vec{F}$. Thus $vw \in F := \{e : \vec{e} \in \vec{F}\}$ is an x,y-disconnecting set. So $\kappa'_G(x,y) \le |F| \le |\vec{F}| = \vec{\kappa}'_{\overleftrightarrow{G}}(x,y)$. □

Theorem 88 (Menger 1927, directed-vertex). *Every directed graph $\vec{G}(V,\vec{E})$ satisfies $\vec{\lambda}(x,y) = \vec{\kappa}(x,y)$ for all distinct $x,y \in V$ with $x\vec{y} \notin \vec{E}$.*

Proof. Let $\vec{H} = (V' \cup V'', \{v'\vec{w}'' : v\vec{w} \in \vec{E}\} \cup \{v''\vec{v}' : v \in V\})$, where $V' = \{v' : v \in V\}$, $V'' = \{v'' : v \in V\}$, and $V' \cap V'' = \emptyset$. For a directed path $\vec{P} = v_1v_2\dots v_s$ in $\vec{G}$ let $\vec{P}' = v''_1v'_1v''_2v'_1\dots v''_{s-1}v'_{s-1}v''_sv'_s$. Then $\vec{P}'$ is a directed path in $\vec{H}$. Every directed v''_1, v'_s-path in $\vec{H}$ has the form of $\vec{P}'$ for a unique path $\vec{P} = v_1\dots v_s$. So $v''v' \in E(\vec{P}')$ if and only if $v \in V(\vec{P})$. It suffices to prove

$$\vec{\kappa}'_{\vec{H}}(x'',y') =_{(0)} \vec{\lambda}'_{\vec{H}}(x'',y') \le_{(1)} \vec{\lambda}_{\vec{G}}(x,y) \le_{(2)} \vec{\kappa}_{\vec{G}}(x,y) \le_{(3)} \vec{\kappa}'_{\vec{H}}(x'',y').$$

Theorem 86 implies (0). As every directed x,y-path meets every directed x,y-separating set, (2) holds.

For (1), let $\vec{\mathcal{P}}' = \{\vec{P}'_i : i \in [l]\}$ be a set of $l := \vec{\lambda}'_{\vec{H}}(x'',y')$ edge-disjoint, directed x,y-paths in $\vec{H}$. Then $\vec{\mathcal{P}} = \{\vec{P}_i : i \in [l]\}$ is a set of $\vec{\lambda}'_{\vec{G}}(x,y)$ independent, directed x,y-paths in $\vec{G}$. So (1) hods.

For (3), let $\vec{F}$ be a directed, x'',y'-disconnecting set in $\vec{H}$ of size $\vec{\kappa}'_{\vec{H}}(x'',y')$. Suppose $\vec{e} \in \vec{F}$. As $x\vec{y} \notin \vec{E}$, if $\vec{e} = v'w''$ then either $v' \neq x'$ or $w'' \neq y''$. In the first case replace $\vec{e}$ by $\vec{e}' := v''\vec{v}'$; else replace $\vec{e}$ by $\vec{e}' := w''\vec{w}'$. Call the resulting set $\vec{F}'$. Then $|\vec{F}'| \le |\vec{F}|$, and $\vec{F}'$ is also a directed x',y''-disconnecting set in $\vec{H}$. Set $S = \{v \in V : v''\vec{v}' \in \vec{F}'\}$. Suppose $\vec{P}$ is a directed x,y-path in G. Then there is $v''\vec{v}' \in \vec{E}(\vec{P}') \cap \vec{F}'$, and $v \in V(\vec{P}) \cap S$. So S is a directed x,y-separating set in $\vec{G}$. Now $\vec{\kappa}_{\vec{G}}(x,y) \le |S| \le |F'| \le |F| = \vec{\kappa}'_{\vec{H}}(x'',y')$, so (2) holds. □

Theorem 89 (Menger 1927, vertex 4.2.17). *Every graph $G(V,E)$ satisfies $\lambda(x,y) = \kappa(x,y)$ for all distinct $x,y \in V$ with $xy \notin E$.*

Proof. It suffices to prove

$$\vec{\kappa}_{\overleftrightarrow{G}}(x,y) =_{(0)} \vec{\lambda}_{\overleftrightarrow{G}}(x,y) \le_{(1)} \lambda_G(x,y) \le_{(2)} \kappa_G(x,y) \le_{(3)} \vec{\kappa}_{\overleftrightarrow{G}}(x,y).$$

Theorem 88 implies (0). As every x,y-path meets every x,y-separating set, (2) holds. For (1) note that if $\vec{P}$ and $\vec{Q}$ are independent, directed x,y-paths then P and Q are independent

x, y-paths. For (3) suppose S is a directed x, y-separating set in $\overleftrightarrow{G}$, and consider an x, y-path $P \subseteq G$. As $\overrightarrow{P}$ is a directed x, y-path, $V(\overrightarrow{P}) \cap S \neq \emptyset$, so S is an x, y-separating set and (3) holds. □

DEFINITION 90. Let $G = (V, E)$ be a graph and A and B be subsets of V. An A, B-*path* is a path with exactly one end in A and exactly one end in B, and whose internal vertices are in neither A nor B. Let $l(A, B)$ be the maximum size of a set of disjoint A, B-paths. An A, B-*separating set* is a set of vertices S such that $G - S$ has no A, B-paths (possibly $S \cap A \neq \emptyset$ or $S \cap B \neq \emptyset$). Let $k(A, B)$ be the minimum cardinality of an A, B-separating set.

THEOREM 91 (Menger 1927 4.2.17). *Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. Then $l(A, B) = k(A, B)$.*

PROOF. For new vertices $a, b \notin V$, let $G^+ = G \cup K(a, A) \cup K(b, B)$. Then $l_G(A, B) = \lambda_{G^+}(a, b) = \kappa_{G^+}(a, b) = k_G(A, B)$. □

THEOREM 92 (4.2.21). *Every graph $G = (V, E)$ satisfies $\kappa(G) = t := \min_{a,b \in V, a \neq b} \lambda(a, b)$.*

PROOF. Choose a, b with $t = \lambda(a, b)$. If G is complete then

$$t = \lambda(a, b) = 1 + (|G| - 2) = |G| - 1 = \kappa(G),$$

since ab is an a, b-path, and acb is also an a, b-path for all $c \in V - a - b$.

Otherwise G has a separating set S with $|S| = \kappa(G)$. Let x, y be vertices in distinct components of $G - S$. Then

$$t = \lambda(a, b) \leq \lambda(x, y) \leq |S| = \kappa(G). \tag{3.8.1}$$

If $ab \notin E$ then by Theorem 89, G has an a, b-separating set U with $t = |U| \geq \kappa(G) \geq t$, and we are done. So assume $ab \in E$, and set $G' = G - ab$. It suffices to show

$$t \leq_{(3.8.1)} \kappa(G) \leq_{(1)} \kappa(G') + 1 \leq_{(2)} \kappa_{G'}(a, b) + 1 =_{(3)} \lambda_{G'}(a, b) + 1 =_{(4)} \lambda_G(a, b) = t.$$

By definition, (2) holds, Theorem 89 implies (3), and (4) holds since ab is an a, b-path.

For (1), suppose $\kappa(G') < \kappa(G)$. Then G' has a separating set W with $|W| = \kappa(G')$, and W is not a separating set of G. It follows that $G' - W$ has exactly two components, say X and Y with $|X| \geq |Y|$, $a \in V(X)$ and $b \in V(Y)$. If $|X| > 1$ then $W + a$ is a separator of G with size $\kappa(G') + 1 \geq \kappa(G)$. Else $|G| = |W| + 2$, and so $\kappa(G) \leq |G| - 1 = \kappa(G') + 1$. □

DEFINITION 93 (4.2.22). Let $G = (V, E)$ be a graph with $x \in V$ and $U \subseteq V$. An x, U-fan is a set $\mathcal{F}$ of x, U-paths such that $|\mathcal{F}| = |U|$ and $F \cap F' = \{x\}$ for all $F, F' \in \mathcal{F}$.

THEOREM 94 (4.2.23). *A graph $G = (V, E)$ is k-connected if and only if $|G| \geq k + 1$ and G has an x, U-fan for all $x \in V$ and all k-sets $U \subseteq V - x$.*

PROOF. (Sketch) Suppose G is k-connected, and consider $x \in V$ and $U \in \binom{V-x}{k}$. Let $u \notin V$ be a new vertex. By Lemma 74, $G^+ := G \cup K(u, U)$ is k-connected. Thus there exist k independent u, x-paths in G^+. Deleting u yields an x, U-fan in G.

Conversely, the hypothesis implies $\delta(G) \geq k$, and for all x and y, there exist k disjoint $N(x), N(y)$-paths. Thus by Theorem 92

$$\kappa(G) = \min_{x,y \in V, x \neq y} \lambda(x, y) \geq k.$$

□

THEOREM 95 (HW 4.2.24). *Let $G = (V, E)$ be a k-connected graph with $k \geq 2$. Then for any k-set $S \subseteq V$ there is a cycle $C \subseteq G$ with $S \subseteq V(C)$.*

PROOF. Let $C \subseteq G$ be a cycle containing as many vertices of S as possible. It exists because $\delta(G) \geq \kappa(G) \geq 2$. We claim that $S \subseteq V(C)$. Otherwise, let $v \in S \smallsetminus V(C)$. Then $|S \cap V(C)| < k$. Arguing by contradiction, it suffices to find a cycle containing $S \cap V(C) + v$.

Orient C cyclically as $\overrightarrow{C}$. Let $t = \min\{k, |C|\}$. By Menger's Theorem there is a set P *of* t disjoint $C, N(v)$-paths; together with v they form a fan $\mathcal{F}$. Set $F = \bigcup \mathcal{F}$, let $x_1, \ldots, x_t$ be a sequence of the leaves of F in cyclic order around $\overrightarrow{C}$, and set $\overrightarrow{C}_i = x_i \overrightarrow{C} x_{i+1}$. Then there exists $i \in [k]$ such that $\overrightarrow{P}_i$ has no internal vertices of S: If $t = |C|$ this is true for all $i \in [k]$; otherwise it follows by the pigeonhole principle, since $t = k > |S \cap V(C)|$. So $D = x_{i+1} \overrightarrow{C} x_i F v F x_{i+1}$ is a cycle containing $S \cap V(C) + v$. □

HW 45. (*) Every 2-connected graph G has a cycle of length at least $\min\{|G|, 2\kappa(G)\}$.

HW 46. (*) If G is a 2-connected graph with $\alpha(G) \leq \kappa(G)$ then G is hamiltonian.

HW 47. (+) Let G be a 2-connected graph that does not induce $K_{1,3}$. Then G has a cycle of length at least $\min\{|G|, 4\kappa(G)\}$.

HW 48. (++) The Conjecture 82 is known to be true in the case $k = 2$ with $f(2) \geq 5$. Prove this with $f(2) \geq 8$. [Hint: Choose P so that $G - P$ contains the biggest 2-connected subgraph possible.]

HW 49. (*) Let $G = (V, E)$ is a graph with $x \in V$ and $Y, Z \subseteq V$, $|Y| = 4$, and $|Z| = 5$. Suppose $\mathcal{Q} = \{Q_y : y \in Y\}$ is an x, Y-fan in G, where each Q_y is an x, y-path. Similarly, suppose $\mathcal{R} = \{R_z : z \in Z\}$ is an x, Z-fan in G, where each R_z is an x, z-path. Prove: There exists a $x, (Y + z)$-fan in G for some $z \in Z$. [Hint: Add new vertices w and x' with neighborhoods $N(w) = Z$ and $N(x') = Y + w$; then apply a theorem.]

CHAPTER 4

Graph coloring

DEFINITION 96. Let $G = (V, E)$ be a graph and C be a set (of colors). A *proper C-coloring* of G is a function $f : V \to C$ such that for all vertices $x, y \in V$ if $xy \in E(G)$ then $f(x) \neq f(y)$. If k is a positive integer, we say that f is a proper k-coloring if it is a proper $[k]$-coloring. The chromatic number $\chi(G)$ is the least k such that G has a proper k-coloring. In this case G is said to be k-chromatic. If G has a k-coloring then it is said to be k-colorable. In this chapter we will assume that all colorings are proper unless otherwise stated. For $i \in C$, $f^{-1}(i)$ is called a color class.

PROPOSITION 97 (HW). *Every graph G satisfies $\omega(G), \frac{|G|}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$.*

HW 50. (*) Prove Proposition 97.

4.1. Examples

EXAMPLE 98 (5.2.3 Mycielski [1955]). For every positive integer k there exists a graph G_k with $\omega(G_k) \leq 2$ and $\chi(G_k) = k$.

PROOF. We argue by induction on k. For $k \leq 2$ let $G_k = K_k$. Now suppose $k \geq 3$ and we have constructed $G_{k-1} = (V_{k-1}, E_{k-1})$ as required. We first construct $G_k = (V_k, E_k)$ as follows: Let $V'_{k-1} = \{v' : v \in V_{k-1}\}$ be a set of new vertices, x_k be a new vertex, and put

$$V_k = V_{k-1} \cup V'_{k-1} + x_k \text{ and,}$$

$$E_k = E_{k-1} \cup \{uv' : uv \in E_{k-1}\} \cup \{x_k v' : v' \in V'_{k-1}\}.$$

So $N(v') \cap V_{k-1} = N(v) \cap V_{k-1}$ for all $v \in V_{k-1}$.

Suppose $\omega(G_k) \geq 3$, and choose $Q = K_3 \subseteq G_k$. Then $k \geq 3$. Since $N(x_k) = V'_{k-1}$ is independent, $x_k \notin Q$, and $|V'_{k-1} \cap Q| \leq 1$. As $\omega(G_{k-1}) =_{i.h.} 2$, there is at least one $v' \in V'_{k-1} \cap Q$. Hence $N(v') \cap Q = N(v) \cap Q$, and $Q - v' + v$ is a K_3 in G_{k-1}, a contradiction.

Now $\chi(G_k) \leq k$: If $k \leq 2$ this is obvious; else G_{k-1} has a $(k-1)$-coloring f' by induction. Extend f' to a k-coloring f of G_k by setting $f(v') = k$ (new color), and $f(x_k) = 1$.

Finally, $\chi(G_k) \geq k$: If $k \leq 2$ this is obvious. If $k \geq 3$ then it suffices to show that every $(k-1)$-coloring g of $G_k - x_k$ satisfies $g(V'_{k-1}) = [k-1]$, since x_k will require a new color. Suppose not. After possibly renaming color classes, let $k - 1 \in [k-1] \smallsetminus g(V'_{k-1})$. For a contradiction, we construct a $(k-2)$-coloring h of G_{k-1}. Define $h : V_{k-1} \to [k-2]$ by:

$$h(v) = \begin{cases} g(v) & \text{if } g(v) \neq k-1 \\ g(v') & \text{if } g(v) = k-1 \end{cases}.$$

If $uv \in E_{k-1}$ then $g(u) \neq g(v)$ since g is proper; if $g(u) \neq k-1 \neq g(v)$ then

$$h(u) = g(u) \neq g(v) = h(v);$$

else exactly one of u, v is colored with $k-1$ by g; say $g(v) = k-1$. Since $uv' \in E_k$,

$$h(u) = g(u) \neq g(v') = h(v). \qquad \square$$

HW 51. (*) Let G_k be the graph in Example 98. Prove that G_k is critical, i.e., $\chi(G_k - e) < \chi(G_k)$ $(= k)$ for all $e \in E_k$. [Hint: There are three types of edges to consider.]

HW 52. (*) Let G be graph, and suppose any two odd cycles $C, C' \subseteq G$ have a common vertex. Prove that $\chi(G) \leq 5$.

HW 53. (*) Let $P = \{v_1, \dots, v_n\}$ be a path, and suppose $G = (V, E)$ is a graph such that V is a subset of the set of subpaths of P, and $E = \{RQ \in V : R \cap Q \neq \emptyset\}$. Prove that $\chi(G) = \omega(G)$. [Hint: Order V so that if $Q, R \in V$ (subpaths of P), the first vertex of Q is v_i, the first vertex of R is v_j, and $i < j$ then Q precedes R, and color V in this order.]

HW 54. (*) Let $G = (V, E)$ be a k-colorable graph, and let P be a set of vertices such that the distance $d_G(x, y)$ between any two points in P is at least 4. Prove that any $[k+1]$-coloring of $G[P]$ can be extended to a $[k+1]$-coloring of G.

4.2. Brooks' Theorem

PROPOSITION 99. *Let $G = (V, E)$ be a graph with $V = A \cup B$. If $A \cap B$ is a separating set and $G[A \cap B]$ is complete, then $\chi(G) = k := \max\{\chi(G[A]), \chi(G[B])\}$.*

PROOF. Let $A \cap B := \{v_1, \dots v_l\}$ where $l = |A \cap B|$. As $A \cap B$ is a clique, $l \leq \omega(G)$. As $V = A \cup B$ and $A \cap B$ is a separating set, every clique of G is contained in A or B. Thus $\omega(G) \leq \max\{\omega(G[A]), \omega(G[B])\} \leq k$. As $\chi(G[A]), \chi(G[B]) \leq k$, there are partitions, $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ (allowing empty sets) of A and B, respectively, into k cocliques. Each coclique contains at most one vertex of $A \cap B$. So we may assume $A_i \cap B_i = \{v_i\}$. Put $C_i := A_i \cup B_i$, and note that C_i is a coclique, since $C_i \cap A \cap B = \{v_i\}$ and $A \cap B$ is a separating set. Then $\{C_1, \dots, C_k\}$ is a partition of V into k cocliques. So $\chi(G) = k$. $\square$

LEMMA 100. *Let $G = (V, E)$ be a connected graph with $v \in V$. There is an ordering $v_1, \dots, v_{|G|}$ of V such that (*) $v = v_{|G|}$ and for all $i \in [|G| - 1]$ there is $j \in [|G|] \setminus [i]$ with $v_i v_j \in E$.*

PROOF. Argue by induction on $|G|$. The base case $n = 1$ is trivial; so assume $n > 1$. Let P be a maximum path; as G is connected and $|G| \geq 2$, $|P| \geq 2$. Pick an end u of P with $u \neq v$. As u is not a cut-vertex, $G - u$ is connected. By induction, there exists an ordering $v_2, \dots, v_n$ of $V - u$ satisfying (*) for $G - u$. So $v_1 := u, v_2, \dots, v_n$ satisfies (*) for G. $\square$

Define a *b-obstruction* to be K_b, or, if $b = 3$, an odd cycle, and let (non-standard) $\omega^*(G)$ be the largest integer b such that G contains a b-obstruction. Then $\omega^*(G) \leq \chi(G) \leq \Delta(G) + 1$.

The following notation will be useful. Recall that for a graph $G = (V, E)$ and subsets $A, B \subseteq V$, $\|A, B\| = \|A, B\|_G = |E_G(A, B)|$. If $A = \{a\}$ or $H \subseteq G$ we may also write $\|a, B\| = \|\{a\}, B\|$ or $\|A, H\| = \|A, V(H)\|$.

THEOREM 101 (Brooks (1941)). *Every graph satisfies $\chi(G) \leq \max\{\omega^*(G), \Delta(G)\}$.*

PROOF. Set $\Delta := \Delta(G)$, $\chi := \chi(G)$, $\omega^* := \omega^*(G)$, and argue by induction on $|G|$. Since $\chi \leq \Delta + 1$, it suffices to show that $\omega^* = \Delta + 1$ or $\chi \leq \Delta$. If $\Delta \leq 1$ then $\omega^* = \Delta + 1$. If $\Delta = 2$ then $2 \leq \omega^* \leq \chi \leq 3$; if $\omega^* = 2$ then $\chi \leq 2$ as G has no odd cycle. So assume $3 \leq \omega^* \leq \Delta$.

By Proposition 99, assume G is 2-connected. Let S be a maximal independent set, and put $G' := G - S$. Then $\Delta(G') < \Delta$, since every vertex of G' has a neighbor in S. If $\omega^*(G') < \Delta$ then $(\Delta - 1)$-color G' by induction, and use a new color for S. Else, consider a Δ-obstruction $Q \subseteq G'$. Choose $y \in S$ with $\|y, Q\| \geq 1$. Since either $Q = K_\Delta$ and $\omega(G) \leq \Delta$ or $\Delta = 3$ and Q is an odd cycle on more than Δ vertices, $V(Q) \nsubseteq N(y)$. As Q is connected, there is an edge $wx \in E(Q)$ with $x \in N(y)$ and $w \notin N(y)$; let $w' \in N_Q(x) - w$.

As G is 2-connected, there is a minimum y, Q-path $P := y \ldots z$ in $G - x$. If $z \neq w$ then set $w^* = w$. If $z = w$ then $\|P\| \geq 2$, and so $yw' \notin E$; set $w^* = w'$. Anyway $z \neq w^*$, $w^*x \in E$, and $w^*y \notin E$. Since $Q - w^*$ and $P - y$ are connected through z, Lemma 100 implies $H := G[Q \cup P]$ has an ordering $L := w^*, y, v_1, \ldots, v_t, x$, where each v_i has a neighbor to its right. Using induction, Δ-color $H' := G - H$ by f. Since

$$\|w^*, H'\| + \|y, H'\| \leq 2\Delta - \|w^*, H\| - \|y, H\| \leq 2\Delta - (\Delta - 1) - 2 \leq \Delta - 1,$$

some color β is not used on $N(w^*) \cup N(y)$. Extend f to G by setting $f(w^*) = \beta = f(y)$ and coloring the remaining vertices in the order L. This is possible, since each v_i has at most $\Delta - 1$ colored neighbors when colored, and x has two neighbors w^*, y colored the same. □

Conjecture 102 (Borodin & Kostochka 1977). *If a graph G satisfies* $8, \omega(G) < \Delta(G)$ *then* $\chi(G) < \Delta(G)$.

Reed used sophisticated methods to prove the conjecture for $\Delta(G) > 10^{14}$.

HW 55. (*) For a graph G let $\theta(G) = \max_{uv \in E(G)}(d(u)+d(v))$. Prove that if $\theta(G) \leq 2r+1$ then $\chi(G) \leq r + 1$. Also prove that if $\theta(G) \leq 2r$ and $\omega^*(G) \leq r$ then $\chi(G) \leq r$.

4.3. Turán's Theorem

Let $n, s \in \mathbb{Z}^+$. In this section we determine the number of edges a graph on n vertices must have to ensure it contains K_s. In other words, how many edges can we put into a graph on n vertices without getting K_s.

Definition 103. A graph is said to be *r-partite* if it is r-colorable. Saying r-partite instead of r-colorable tends to emphasize the partition into r independent sets provided by the r-coloring. These independent sets are called *parts.* The *complete r-partite* $K_{n_1,\ldots n_r}$ graph is the r-partite graph with r parts of sizes $n_1, \ldots, n_r$ such that any two vertices in different parts are adjacent. The *Turán* graph $T_{n,r}$ is the complete r-partite graph on n vertices such that any two parts differ in size by at most one.

Lemma 104 (5.2.8). *Among all r-partite graphs on n vertices, $T_{n,r}$ has the most edges.*

Proof. Let G be an r-partite graph on n vertices with as many edges as possible; say $\mathcal{X}$ is an r-partition of G. Clearly, G is a complete r-partite graph. So, if $G \neq T_{n,r}$ then there exist parts $X, Y \in \mathcal{X}$ with $|X| - |Y| \geq 2$ and $x \in X$. Let G' be the complete r-partite graph with r-partition $\mathcal{X}' := \mathcal{X} - X - Y + (X - x) + (Y + x)$. Then

$$E(G') \supseteq E(G) - \{xy : y \in Y\} + \{xx' : x' \in X - x\}.$$

Thus

$$\|G'\| \geq \|G\| - |Y| + |X| - 1 \geq \|G\| + 1,$$

a contradiction. So $G \cong T_{n,r}$. □

Theorem 105 (5.2.9 Turán [1941]). *Among all graphs $G = (V, E)$ on n vertices with $\omega(G) \leq r$, the one with the most edges is $T_{n,r}$.*

Proof. Evidently $T_{n,r}$ is a candidate. Argue by induction on r that if G satisfies $|G| = n$, $\omega(G) \leq r$, and $\|G\| \geq \|T_{n,r}\|$ then $G \cong T_{n,r}$. If $\omega(G) = 1$ then $G \cong T_{n,1}$; so suppose $r > 1$.

Choose $v \in V$ with $d(v) = \Delta := \Delta(G)$. Set $N := N(v)$, $G' := G[N]$, $S := V - N(v)$ and $G'' := G[S]$. Then $|G'| = \Delta$, and $\omega(G') \leq r - 1$, since $K + v$ is a clique in G for every clique K in G'. Set $H := T_{\Delta,r-1} \vee \overline{K}(S)$. Then H is an r-partite graph on n vertices, and $\omega(H) \leq r$, since any clique in H has at most $r-1$ vertices in $T_{\Delta,r-1}$ and one vertex in S. So

$$\begin{aligned}
\|G\| &= \|G'\| + \|G''\| + |E(N,S)| \\
&= \|G'\| + \sum_{v \in S} d_G(v) - \|G''\| \\
&\leq \|T_{\Delta,r-1}\| + \sum_{v \in S} d_G(v) - \|G''\| && \text{(induction)} \quad (4.3.1)\\
&\leq \|T_{\Delta,r-1}\| + \Delta|S| && \text{(maximum degree)} \quad (4.3.2)\\
&= \|H\| \\
&\leq \|T_{n,r}\| && \text{(Lemma 104)} \quad (4.3.3)
\end{aligned}$$

Inequality (4.3.1) is strict unless $G' \cong T_{\Delta,r-1}$. Inequality (4.3.2) is strict unless $G'' = \overline{K}(S)$ and $G = G' \vee G''$. Inequality (4.3.3) is strict unless $H \cong T_{n,r}$. If $\|G\| \geq \|T_{n,r}\|$ then all three inequalities are tight, and so

$$G \cong T_{\Delta,r-1} \vee \overline{K}(S) \cong H \cong T_{n,r}.$$

□

HW 56. (*) Prove that if $\omega(G) \leq r$ then $\|G\| \leq (1 - 1/r)|G|^2/2$.

4.4. Edge Coloring

Definition 106. Let $G = (V, E)$ be a graph. A *proper k-edge-coloring* of G is a function $f : E \to [k]$ such that $f(e) = f(e')$ implies that e and e' are not adjacent ($e \cap e' = 0$). The *chromatic index* $\chi'(G)$ of G is the least k such that G has a proper k-edge-coloring. In this section we will assume that all edge colorings are proper. Note that this is not the case when we consider Ramsey Theory.

HW 57. (*) Let P be the Petersen graph and $v \in V(P)$. Determine $\chi'(P - v)$.

Theorem 107 (7.1.17 König [1916]). *Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.*

Proof. Argue by induction on $\Delta = \Delta(G)$. The base step $\Delta = 1$ is trivial since G has no adjacent edges, and so all edges can receive the same color. So consider the induction step $\Delta > 1$. It suffices to find a Δ-regular bipartite multigraph H with $G \subseteq H$: By Theorem 62, H has a perfect matching M. Color all edges in $M \cap E(G)$ with color Δ, and set $G' = G - M$. Then $\Delta(G') \leq \Delta(H - M) \leq \Delta - 1$, and so by induction, G' has a $(\Delta - 1)$-edge-coloring. Using another color on M yields a Δ-edge-coloring of G.

It remains to construct H. Suppose G is an A, B-bigraph with $|A| \leq |B|$. Let $A \subseteq A'$, where $A' \cap B = \emptyset$ and $|A'| = |B|$. Choose an A', B-bipartite multigraph $H \supseteq G$ with $\|H\|$

maximal subject to $\Delta(H) \leq \Delta$. It exists because G is a candidate. Now

$$\|H\| = \sum_{v \in A'} d_{G'}(v) = \sum_{v \in B} d_{G'}(v) \leq \Delta|B|.$$

If $\|H\| = \Delta|B|$ then H is Δ-regular. Else there exist vertices $a \in A'$ and $b \in B$ with $d_H(a), d_H(b) < \Delta$. Then $H' := H + e$, where $e \in E(a, b)$ is a new, possibly parallel edge, contradicts the maximality of H. □

Now we consider edge coloring of general graphs. The fundamental result is Theorem 109 due to Vizing. The following lemma does most of the work in its proof.

LEMMA 108. *Suppose $G = (V, E)$ is a simple graph with $\Delta(G) \leq k \in \mathbb{N}$, and $v \in V$. If $\chi'(G - v) \leq k$ and $d(x) = k$ for at most one $x \in N(v)$ then $\chi'(G) \leq k$.*

PROOF. Argue by induction on k. If $k = 1$ then E is a matching, and so $\chi'(G) \leq 1$. Now suppose $k > 1$. For a function $f : E \to [k]$, $x \in V$ and $\alpha \in [k]$, set

$$f(x) := [k] \smallsetminus \{f(e) : e \in E(x)\} \text{ and } f_\alpha := \{u \in N(v) : \alpha \in f(u)\}.$$

By adding edges and vertices to G, we may assume $k-1 \leq d(x) \leq k = d(v)$ for all $x \in N(v)$, and $d(y) = k$ for exactly one $y \in N(v)$. So $|f(x)| = 2$ for all $x \in N(v) - y$ and $|f(y)| = 1$. Pick a k-edge-coloring f of $G' := G - v$ with $T(f) := \{\beta \in [k] : 1 \leq |f_\beta| \leq 2\}$ maximum.

Suppose $|f_\alpha| \neq 1$ for all $\alpha \in [k]$. Since $\sum_{\alpha \in [k]} |f_\alpha| = \sum_{x \in N(v)} |f(x)| = 2k - 1$, there exist $\beta, \gamma \in [k]$ with $|f_\beta| = 0$ and $|f_\gamma| \geq 3$; say $w \in f_\gamma$. Set $G_{\beta,\gamma} = (V - v, E_{\beta,\gamma})$, where $E_{\beta,\gamma} = \{e \in E : f(e) \in \{\beta, \gamma\}\}$. Then the component of $G_{\beta,\gamma}$ containing w is a path P with ends w and (say) z, where $f(z) \cap \{\beta, \gamma\} \neq \emptyset$. Switching colors γ and β on the edges of P yields a new coloring f' of G'. Then $f'(u) = f(u)$ for $u \in V(G') \smallsetminus \{w, z\}$, and $f'(w) = f(w) - \gamma + \beta$. So $f'_\gamma \neq \emptyset$ and $w \in f'_\beta \subseteq \{w, z\}$. Thus $T(f) \subsetneqq T(f') + \beta$, contradicting the choice of f.

So $f_\alpha = \{z\}$ for some $\alpha \in [k]$ and $z \in N(v)$; say $\alpha = k$. Set $M = f^{-1}(k) + vz$. Since neither z nor v are incident to any edges colored k, M is a matching. Put $H = G - M$. Since $f_k = \{z\}$ and $vz \in M$, every vertex of $N[v]$ is M-saturated. So $d_H(x) \leq k-1$ for every $x \in N_H(v)$, and equality holds at most once. Since $f^{-1}(k) \subseteq M$, f is a $(k-1)$-coloring of $H - v$, and $\Delta(H - v) \leq k - 1$. So $\Delta(H) \leq k - 1$. By induction, $\chi'(G) \leq \chi'(H) + 1 \leq_{i.h.} k$. □

THEOREM 109 (7.1.10 Vizing (1964)). *Every graph $G = (V, E)$ satisfies $\chi'(G) \leq \Delta(G) + 1$.*

PROOF. Set $k := \Delta(G) + 1$ and argue by induction on $|G|$. If $|G| = 1$ then $\chi'(G) \leq 1 = k$. Otherwise choose $v \in V$. By induction, $\chi'(G - v) \leq k$, and so by Lemma 108, $\chi'(G) \leq k$. □

THEOREM 110 (Full Vizing (1964)). *Every multigraph M satisfies $\chi'(M) \leq \Delta(M) + \mu(M)$.*

HW 58 (*). Let G be a graph with $\Delta(G) = k$. Put $X = \{v \in V(G) : d(v) = k\}$. Prove that if $G[X]$ is acyclic then $\chi'(G) \leq k$. [Hint: Use Lemma 108.]

HW 59 (+). Prove Theorem 110.

CONJECTURE 111 (Goldberg (1973), Seymour (1979)). *Every multigraph M with $\chi'(M) \geq \Delta(M) + 2$ satisfies $\chi'(M) = \max_{H \subseteq M} \lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor} \rceil$.*

HW 60 (*). Prove that $\chi'(M) \geq \max_{H \subseteq M} \lceil \frac{\|H\|}{\lfloor |H|/2 \rfloor} \rceil$.

DEFINITION 112 (4.2.18). The line graph $H = L(G)$ of a graph $G = (V, E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

If H is the line graph of a simple graph G then H contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ (a K_5 missing one edge). Also, $\chi(H) = \chi'(G)$ and $\omega(H) = \Delta(G)$, unless $\Delta(G) = 2$ and $\omega(G) = 3$. So the following theorem (with an extra observation for the case $\Delta(G) = 2 < \omega(G)$) extends Vizing's Theorem for simple graphs.

THEOREM 113 (Kierstead & Schmerl 1983). *Every graph H that contains neither an induced copy of $K_{1,3}$ nor an induced copy of $K_5 - e$ satisfies $\chi(H) \leq \omega(H) + 1$.*

4.5. List Coloring

DEFINITION 114. Let $G = (V, E)$ be a graph and C a set of colors. We write 2^C for the power set of C. A *list assignment* for G is a function $f : V \to 2^C$. One should think of $f(v) \subseteq C$ as the set of colors that are available for coloring the vertex v. A *k-list assignment* is a list assignment f such that $|f(v)| = k$ for all $v \in V$. Given a list assignment f, an f-coloring is a proper coloring g such that $g(v) \in f(v)$ for all $v \in V$. In this case G is f-colorable. The graph G is *k-list-colorable* (also *k-choosable*) if for every k-list assignment f it is f-colorable. The *list-chromatic number* (also *choosability*, also *choice number*) $\chi_l(G)$ *of* G is the least k such that it is k-list colorable.

EXAMPLE 115. Let $G = K_{t,t^t}$. Then $\chi(G) = 2$, but $\chi_l(G) \geq t + 1$.

PROOF. Let X, Y be a bipartition of G with $|X| = t$. Let f be a t-list assignment for G such that the vertices of X have disjoint lists of size t, and for each $\sigma \in \prod_{x \in X} f(x)$ there exists $y_\sigma \in Y$ with $f(y) = \text{range}(\sigma)$. Then for any f-coloring σ of $G[X]$, the vertex v_σ cannot be colored from the list $f(y_\sigma) = \sigma$. □

DEFINITION 116. An *edge-list assignment* for G is a function $f : E \to 2^C$. One should think of $f(e) \subseteq C$ as the set of colors that are available for coloring the edge e. A *k-edge-list assignment* is a list assignment f such that $|f(e)| = k$ for all $e \in E$. Given an edge-list assignment f, an f-coloring is a proper edge-coloring g such that $g(e) \in f(e)$ for all $e \in E$. In this case, G is f-list-colorable. The graph G is *k-edge-list-colorable* (also *k-edge-choosable*) if for every k-edge-list assignment f, it is f-colorable. The *list-chromatic index* (also *edge-choosability, edge-choice number*) $\chi'_l(G)$ *of* G is the least k such that it is k-edge-list colorable.

CONJECTURE 117. *Every graph G satisfies $\chi'_l(G) = \chi'(G)$.*

DEFINITION 118. A *kernel* of a digraph $D = (V, A)$ is an independent set $S \subseteq V$ such that for every $x \in V \setminus S$ there exists $y \in S$ with $xy \in A$.

The 5-cycle $G = (V, E) = v_1v_2v_3v_4v_5v_1$ has several orientations.

$$E_1 = \{v_1\vec{v}_2, v_2\vec{v}_3, v_3\vec{v}_4, v_4\vec{v}_5, v_5\vec{v}_1\} \text{ and } E_2 = \{v_1\vec{v}_2, v_3\vec{v}_2, v_3\vec{v}_4, v_5\vec{v}_4, v_5\vec{v}_1\}.$$

$\vec{G}_1 := (V, \vec{E}_1)$ does not have a kernel since $\Delta^+(\vec{G}_1) = 1$, $\alpha(G) = 2$, and $|\vec{G}| = 5 > \alpha(G) \cdot (\Delta^+(\vec{G}_1) + 1)$, but $\{v_2, v_4\}$ is a kernel of $\vec{G}_2 := (V, \vec{E}_2)$.

LEMMA 119 (8.4.29 Bondy & Boppana & Siegel). *Let $D = (V, A)$ be a digraph all of whose induced subgraphs have kernels. If f is a list assignment for D satisfying $d^+(v) < |f(v)|$ for all $v \in V$ then D has an f-coloring.*

PROOF. Argue by induction on $|D|$. Let $v_0 \in V$. Since $|f(v_0)| > d^+(v_0) \geq 0$, there is $\alpha \in f(v_0)$. Set $W = \{v \in V : \alpha \in f(v)\}$. Then $v_0 \in W$. By hypothesis $D[W]$ has a (nonempty) kernel S. Color every vertex in S with α. This is possible because S is independent and $S \subseteq W$.

Now it suffices to f-color $D' = D - S$ so that no vertex in $D - S$ is colored α. For this purpose, let f' be the list assignment for D' defined by $f'(v) = f(v) - \alpha$. Since $|D'| < |D|$, using induction, it suffices to show $|f'(v)| > d^+_{D'}(v)$ for all $v \in V \smallsetminus S$.

If $v \notin W$ then $\alpha \notin f(v)$, and so

$$|f'(v)| = |f(v)| > d^+_D(v) \geq d^+_{D'}(v).$$

Else $v \in W$. Since S is a kernel of $D[W]$, there exists $w \in S = V \smallsetminus V(D')$ with $vw \in A$. So

$$|f'(v)| = |f(v) - \alpha| > d^+_D(v) - 1 \geq d^+_{D'}(v). \qquad \square$$

THEOREM 120 (8.4.30 Galvin 1995). *Every X, Y-bigraph G satisfies $\chi'_l(G) = \Delta(G)$.*

PROOF. Let $\Delta := \Delta(G)$ and set $H := L(G)$. Then $\chi'_l(G) = \chi_l(H)$ and $\chi'(G) = \Delta$ (Theorem 107). Fix a Δ-edge coloring $c : E(G) \to [\Delta]$.

Let L be a Δ-edge-list assignment for G; so L is a Δ-list assignment for H. Our plan is to apply Lemma 119 to H to show that it has an L-coloring f; then f is an L-edge-coloring of G. So it suffices to show H has an orientation $D := (\vec{E}(G), A)$ such that (i) $\Delta^+(D) \leq \Delta - 1$ and (ii) every induced subgraph $D' \subseteq D$ has a kernel.

Each $\vec{ee'} \in E(H)$ satisfies $e \cap e' \subseteq X$ or $e \cap e' \subseteq Y$. Define D by putting

$$\vec{ee'} \in A \text{ iff } (e \cap e' \subseteq X \wedge c(e) > c(e')) \vee (e \cap e' \in Y \wedge c(e) < c(e')).$$

Each $e \in E(G) = V(H)$ satisfies $d^+_H(e) \leq \Delta - 1$, since it has at most $c(e) - 1$ out-neighbors e' with $e \cap e' \subseteq X$ and $\Delta - c(e)$ out-neighbors e' with $e \cap e' \subseteq Y$. So (i) holds for D.

For (ii), let $D' = D[F]$, where $F \subseteq E(G) = V(D)$. Argue by induction on $|D'|$. Let $X' := \{x \in X : E_G(x) \cap F \neq \emptyset\}$. For each $x \in X'$, choose $e_x \in E_G(x) \cap F$ with $c(e_x)$ minimum. Then $\vec{ee_x} \in A$ for every $e \in E_G(x) \cap F - e_x$. If $Q = \{e_x : x \in X'\}$ is independent then it is a kernel of D'; else fix distinct $x, x' \in X'$ with $e_x \cap e_{x'} \neq \emptyset$. Then $e_x, e_{x'} \in E_G(y)$ for some $y \in Y$; say $c(e_x) < c(e_{x'})$. Let $D'' = D' - e_x$. By induction, D'' has a kernel S. If $e_{x'} \in S$ then S is a kernel for D', since $\vec{e_xe_{x'}} \in A$. Otherwise, $\vec{e_{x'}e^*} \in A$ for some $e^* \in S$. The choice of $e_{x'}$ implies $e_{x'} \cap e^* \nsubseteq X$. So $e_{x'} \cap e^* \subseteq Y$, and thus $e_x, e_{x'}, e^* \in E_G(y)$. So $c(e^*) > c(e_{x'}) > c(e_x)$; so $\vec{e_xe^*} \in A$. Hence S is a kernel for D'. $\square$

CHAPTER 5

Planar graphs

We have been informally drawing graphs in the Euclidean plane $\mathbb{R}^2$ since the start of the semester. Now we formalize the definition of a drawing of a graph in $\mathbb{R}^2$.

5.1. Very Basic Topology of the Euclidean Plane

For $x \in \mathbb{R}^2$ the *open ball around x with radius r* is the set $B_r(x) := \{y \in \mathbb{R}^2 : \|x, y\| < r\}$. A set $U \subseteq \mathbb{R}^2$ is *open* if for all points $p \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$. In particular, $\mathbb{R}^2$ and $\emptyset$ are open. The complement of an open set is a *closed* set. The *frontier* of a set X is the set of all points $x \in \mathbb{R}^2$ such that $B_r(x) \cap X \neq \emptyset$ and $B_r(x) \smallsetminus X \neq \emptyset$ for all $r > 0$. Note that if X is open, then its frontier lies in $\mathbb{R}^2 \smallsetminus X$.

Let $p, q \in \mathbb{R}^2$. The p, q-line segment $L_{p,q}$ is the subset of $\mathbb{R}^2$ defined by $L(p, q) := \{p + \lambda(q - p) : 0 \leq \lambda \leq 1\}$ and $\mathring{L}(p, q) := L(p, q) \smallsetminus \{p, q\}$. For distinct points $p_0, \ldots, p_k \in \mathbb{R}^2$, the union $A(p_0, \ldots, p_k) := \bigcup_{i \in [k]} L(p_{i-1}, p_i)$ is a (polygonal) p_0, p_k-arc provided $L(p_{i-1}, p_i) \cap \mathring{L}(p_{j-1}, p_j) = \emptyset$ for all distinct $i, j \in [k]$. We say that p_0 and p_k are linked by $A(p_0, \ldots, p_k)$. If $A(p_0, \ldots, p_k)$ is an arc and $L(p_0, \ldots, p_k) \cap \mathring{L}(p_k, p_0) = \emptyset$ then $P(p_0, \ldots, p_k, p_0) := L(p_0, \ldots, p_k) \cup L(p_k, p_0)$ is a polygon. Note that arcs and polygons are closed in $\mathbb{R}^2$.

Let U be an open set. Two points $x, y \in U$ are linked in U if there exists an x, y-arc contained in U. The relation of being linked is an equivalence relation on U. Its equivalence classes are called *regions*. Regions are open: Suppose $R \subseteq U$ is a region and $x \in R$. Then there exists a $r > 0$ such that $B_r(x) \subseteq U$. Clearly every $y \in B_r(x)$ is linked to x in U, since $L(x, y) \subseteq B_r(x)$. So $B_r(x) \subseteq R$. A closed set X separates a region R if $R \smallsetminus X$ has more than one region.

THEOREM 121 (Jordan Curve Theorem for Polygons). *For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \smallsetminus P$ has exactly two regions. Each of these regions has the entire polygon as its frontier.*

PROOF. *To be continued ...* □

5.2. Graph Drawings

Recall that in this class all graphs are finite. Let $G = (V, E)$ be a (multi)graph. A *drawing* of G is a graph $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ such that $G \cong \widetilde{G}$, $\widetilde{V} \subseteq \mathbb{R}^2$, and each edge $\tilde{e} \in \widetilde{E}$ is an arc linking its ends. So edges are no longer just pairs of vertices, but have their own identity and structure (we need this anyway to formally deal with different edges linking the same two vertices). It should be clear that every finite graph has a drawing. Moreover, by moving vertices slightly and readjusting edges, we can (and do) require the following additional properties for drawings, without restricting the set of (finite) graphs that can be drawn.

(1) No three edges have a common internal point.

(2) The only vertices contained in an edge are its endpoints.
(3) No two edges are tangent.
(4) No two edges have more than one common internal point.

A crossing in a drawing of a graph is a point that is in the interior of two edges. A *plane* (multi)graph is a drawing of a (multi)graph that has no crossing. A *planar* (multi)graph is a (multi)graph that has a plane drawing.

Let $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ be a plane (multi)graph. The faces of $\widetilde{G}$ are the regions of $\mathbb{R}^2 \setminus (\widetilde{V} \cup \bigcup \widetilde{E})$. The frontier of a face is called its *boundary.* A boundary is the union of some edges and vertices of G—so it is a subgraph of $\tilde{G}$. The edges of a face f can be ordered to form its *boundary walk(s)*; this is a closed walk(s) that contains every edge once or twice. Let $l(f)$ be the length of its boundary walk. Let $F(\widetilde{G})$ be the set of faces of $\widetilde{G}$.

Note that a plane cycle is a polygon. By the Jordan Curve Theorem we have:

PROPOSITION 122 (Top.). *A plane cycle is the boundary of exactly two faces.*

To be continued ...

5.3. Basic facts

FACT 123. *[Top.]Let $\widetilde{G}$ be a plane multigraph with $e \in E$. If e is a cut-edge then $|F(\widetilde{G})-e)| = |F(\widetilde{G})|$; if e is not a cut-edge then $|F(\widetilde{G}-e)| = |F(\widetilde{G})| - 1$.*

FACT 124. *If G is a planar multigraph and $e \in E$ then $G \cdot e$ is a planar graph.*

FACT 125. *[Top.]Let $\widetilde{G}$ be a plane graph with link (non-loop-edge) $e = xy$. Let $\widetilde{G} \cdot e$ be the plane graph formed from $\widetilde{G}$ by deleting y and adding edges xv in the face of $\widetilde{G}$-y containing y for all $v \in N(y) \setminus N(x)$. Then $|F(\widetilde{G} \cdot e)| = |F(\widetilde{G})|$. Deleting a loop of $\widetilde{G}$, reduces the number of faces by 1.*

For an edge e and a face f, let $\iota(e, f)$ be the number of times e appears on the boundary walk of f.

FACT 126. *[Top.]Let $\widetilde{G}$ be a plane multigraph. Every non-cut-edge appears exactly once on the bounding walk of exactly two faces. Every cut-edge appears exactly twice on the bounding walk of exactly one face. In particular,*

$$\sum_{f \in F(\widetilde{G})} l(f) = \sum_{f \in F(\widetilde{G})} \sum_{e \in E(\widetilde{G})} \iota(e, f) = \sum_{e \in E(\widetilde{G})} \sum_{f \in F(\widetilde{G})} \iota(e, f) = 2\left\|\widetilde{G}\right\|.$$

THEOREM 127 (6.1.21 Euler's Formula (1758)). *All connected, planar multigraphs G satisfy*

$$|G| - \|G\| + |F(G)| = 2.$$

PROOF. Argue by induction on $|G|$.

Base Step: $|G| = 1$. In this case all edges of G are loops. Argue by secondary induction on $\|G\|$. For the base step $\|G\| = 0$, note that G has one vertex and one face, and so

$$|G| - \|G\| + |F(G)| = 1 - 0 + 1 = 2.$$

For the induction step, set $G' := G - l$ for some loop l. Then using Fact 125:

$$|G| - \|G\| + |F(G)| = |G'| - (\|G'\| + 1) + (|F(G')| + 1) =_{i.h.} 2.$$

Induction Step: $|G| > 1$. Since G is connected, G has a non-loop edge e. Set $G' = G \cdot e$. By Fact 125,

$$|G| - \|G\| + |F(G)| = (|G'| + 1) - (\|G'\| + 1) + |F(G')| = 2.$$

□

THEOREM 128. *If G is a simple planar graph with at least three vertices then $\|G\| \leq 3|G| - 6$. Moreover, if G has girth greater than 3 then $\|G\| \leq 2|G| - 4$.*

PROOF. We may assume that G is a maximal planar graph, i.e., it is not a spanning subgraph of any planar graph with more edges. Then G is connected, since otherwise we could add an edge between two components of G while maintaining planarity. Since G is connected and has at least three vertices, $\|G\| \geq 2$; since it is simple it has no parallel edges. Thus the length of every face is at least 3. By Fact 126,

$$2\,\|G\| = \sum_{f \in F(G)} l(f) \geq 3|F(G)|.$$

So $|F(G)| \leq \frac{2}{3}\,\|G\|$. By Theorem 127

$$2 = |G| - \|G\| + |F(G)| \leq |G| - \frac{1}{3}\,\|G\|\,,$$

and so $3|G| - 6 \geq \|G\|$.

Now suppose that the girth of G is greater than 3. Then every face boundary has length at least 4. So

$$2\,\|G\| = \sum_{f \in F(G)} l(f) \geq 4|F(G)|.$$

Thus $|F(G)| \leq \frac{1}{2}\,\|G\|$. By Theorem 127

$$2 = |G| - \|G\| + |F(G)| \leq |G| - \frac{1}{2}\,\|G\|\,,$$

and so $2|G| - 4 \geq \|G\|$. □

A graph H is a subdivision of a graph G if H is formed by replacing some of the edges $xy \in E(G)$ by an x,y-path P whose internal vertices have degree 2.

COROLLARY 129. *Neither K_5 nor $K_{3,3}$ is planar.*

PROOF. If K_5 is planar then Theorem 128 yields the following contradiction:

$$10 = \|K_5\| \leq 3|K_5| - 6 = 9.$$

If $K_{3,3}$ is planar, then since it is bipartite, and so has girth greater than 3, Theorem 128 yields the contradiction:

$$9 = \|K_{3,3}\| \leq 2|K_{3,3}| - 4 = 8.$$

□

A graph H is a subdivision of a graph G if H is formed by replacing some of the edges $xy \in E(G)$ by an x, y-path P whose internal vertices are not vertices of G and have degree 2 in H. The vertices of G are called branch vertices and the new vertices are called subdivision vertices. Notice G is not a subgraph of a proper subdivision of itself.

THEOREM 130 (6.2.2 Kuratowski (1930)). *A graph is planar iff it contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.*

We will break the proof of Kuratowski's Theorem into smaller pieces. Call a subdivision of K_5 or $K_{3,3}$ a *K-graph* (for Kuratowski).

LEMMA 131. *Let $e = x_1x_2$ be an edge of a graph G. If G contains no K-graph then $G \cdot e$ contains no K-graph.*

PROOF. We prove the contrapositive. Suppose $G \cdot e$ contains a K-graph Q. If $Q \subseteq G$ we are done; else $v_e \in V(Q)$. Then $3 \le d_Q(v_e) \le 4$ and $N_Q(v_e) \subseteq N_G(x_1) \cup N_G(x_2)$. If there is $i \in [2]$ with $|N_Q(v_e) \smallsetminus N_G(x_i)| \le 1$ then replacing v_e by x_i or x_1, x_2, e yields a subdivision of Q. Otherwise, $d_Q(v_e) = 4$, $|N_Q(v_e) \smallsetminus N_G(x_1)| = 2 = |N_Q(v_e) \smallsetminus N_G(x_2)|$, and Q is a subdivision of $K_5(\{v_e, a, b, c, d\})$. We claim G contains a subdivision of $K_{3,3}$! Let $Q' = Q - v_e + x_1 + x_2 + e$. Then Q' is a subdivision of $Q' = K_{3,3}\{\{x_1, a, b\}, \{x_2, c, d\}\}$. □

COROLLARY 132. *If G is planar then G does not contain a K-graph.*

PROOF. Suppose $Q \subseteq G$ is a K-graph, and show that G is not planar by induction on the number h of subdivision vertices in Q. The base step $h = 0$ is Corollary 129. For the induction step $h > 0$, consider a subdivision vertex x and one of its two neighbors y. Contracting xy yields a K-graph with one less subdivision vertex. Thus by the induction $G \cdot xy$ is nonplanar, and by Fact 124, G is nonplanar. □

Let $G = (V, E)$ be a graph, and $S \subseteq V$. An S-lobe is a subgraph of the form $G[S \cup V(H)]$, where H is a component of $G - S$.

LEMMA 133 (Top.). *Let G be a planar* 2-connected *planar graph with plane drawing $\widetilde{G}$. Then the boundary walk of every face of $\widetilde{G}$ is a cycle.*

PROOF. By Theorem 76, G has an a 2-witness $P_0, \dots, P_h$. Argue by induction on h. For the base step $h = 0$, note that P_0 is a cycle that bounds the only two faces of $\widetilde{G}$ (Jordan Curve Theorem). Now consider the induction step. Let P_h have ends x, y, and note that $\mathring{P}_h$ is contained in some face f of $\widetilde{H} = \widetilde{G} - \mathring{P}_h$. By the induction hypothesis, every face of $\widetilde{H}$ is bounded by a cycle. Let $C = xv_1 \dots v_a y v_{a+2} \dots v_b x$ be the cycle that bounds f. Let f_1 and f_2 be the faces of $\widetilde{G}$ that are contained in f and bounded by xv_1CyP^*x and $yv_{a+2}CxPy$. Then $F(\widetilde{G}) = F(\widetilde{H}) - f + f_1 + f_2$, and all are bounded by cycles. □

THEOREM 134. *If G is a 3-connected graph that does not contain any K-graph then G is planar.*

PROOF. We argue by induction on $|G|$. If $|G| \le 4$ then G is planar, since $G \subseteq K_4$, and K_4 is planar. So suppose $|G| \ge 5$. By Theorem 79, G has an edge $e = xy$ such that $H := G \cdot e$ is 3-connected. By Corollary 131, H does not contain a K-graph. Thus by the induction hypothesis, H is planar. Let $\widetilde{H}$ be a drawing of H.

Let $H' = H - v_e$, and let f be the face of $\widetilde{H} - v_e$ that contains v_e. Since H' is 2-connected, Lemma 133 implies that f is bounded by a cycle C, say with orientation $\overrightarrow{C}$, and $(N_G(x) \cup N_G(y)) \setminus \{x, y\} = N_H(v_e) \subseteq V(C)$. Choose the notation so that $d_G(x) \leq d_G(y)$. Obtain a drawing $\widetilde{G'}$ of $G' := G - y$ from $\widetilde{H}'$ by drawing x at the point corresponding to v_e in $\widetilde{H}$ and deleting the edges $v_e z$ with $z \in N_G(y) \setminus N_G(x)$. Our goal is to extend $\widetilde{G}'$ to a drawing $\widetilde{G}$ of G by adding y and the edges in $E(y)$ to $\widetilde{G}'$.

Let $x_1, \ldots, x_k$ be the neighbors of x in G' arranged in cyclic order around $\overrightarrow{C}$, and $U := N_G(y) - x$. If there exists an index i such that $U \subseteq C_i := V(x_i \overrightarrow{C} x_{i+1})$ then we can extend $\widetilde{G'}$ to $\widetilde{G}$ by drawing y in the face f' of $\widetilde{G'}$ bounded by $x x_i C x_{i+1} x$, and then drawing edges from y to each vertex of $U + x$. This is possible, because all vertices of $U + x$ appear on the boundary of f'. Otherwise for every index i there exists $u_i \in U$ such that $u_i \notin C_i$.

If there exists an index i and a vertex $v_i \in N(y) \cap V(C_i - x_i - x_{i+1})$ then the vertices

$$v_i, u_i, x, x_i, x_{i+1}, y$$

are the branch vertices of a subdivision of $K_{3,3}$ with bipartition $\{\{v_i, u_i, x\}, \{x_i, x_{i+1}, y\}\}$, where the edges of the cycle $x_i v_i x_{i+1} u_i x_i$ are represented by the paths

$$x_i \overrightarrow{C} v_i, v_i \overrightarrow{C} x_{i+1}, x_{i+1} \overrightarrow{C} u_i, u_i \overrightarrow{C} x_i.$$

(Draw the picture.) This is a contradiction.

Otherwise, $U \subseteq N_G(x)$. Since $d_G(x) \leq d_G(y)$, we have $U = N_G(x) - y$. Thus U separates xy from the rest of G. Since G is 3-connected, $|U| \geq 3$. Say $a, b, c \in U$. Then x, y, a, b, c are the branch vertices of a subdivision of a K_5, where the edges of the cycle $abca$ are represented by the paths $a \overrightarrow{C} b, b \overrightarrow{C} c, c \overrightarrow{C} a$. (Draw the picture.) This is a contradiction. □

LEMMA 135. *Let $G = (V, E)$ be a graph with $V = V_1 \cup V_2$ such that $S := V_1 \cap V_2$ is a minimum separating set with $|S| \leq 2$. If G is edge-maximal among graphs without K-graphs then so are $G_1 := G[V_1]$ and $G_2 := G[V_2]$, and $G[S] = K_2$.*

PROOF. Let $S = \{u^i : i \in [|S|]\}$. As S is a minimum separating set, there are $v_j^i \in N(u^i) \cap (V_j \setminus S)$ for all $u^i \in S$ and $j \in [2]$, and $E(V_1 \setminus S, V_2 \setminus S) = \emptyset$. Let $w_i \in V_i \setminus S$ for $i \in [2]$. By maximality $G + e$ contains a K-graph H with $e \in E(H)$ for all $e \in E(\overline{G})$.

If $S = \emptyset$ then $e := w_1 w_2$ is a cut-edge in $G + e$. As K-graphs contain no cut-edges, this is a contradiction. Now suppose $|S| = 1$. Then $G' := G + v_1^1 v_1^2$ contains a K-graph H. As any two branch vertices of H are linked by 3 disjoint paths in H, all the branch vertices are contained in, say, G_1. Thus there is a path $P = v_1^1 v_2^1 \ldots u^1$ with $G'[V_2 + v_1^1] = P$. Replacing P by $v_1^1 u^1$ yields an K-graph in $G_1 \subseteq G$ a contradiction. So $|S| = 2$. Suppose $e := u^1 u^2 \notin E$. Then $G + e$ contains a K-graph H with $e \in E(H)$. Again, the branch vertices of H are contained in (say) G_1. Then $H \subseteq G_1 + e$. Replacing e by a u^1, u^2-path in G_2 yields an K-graph in G, a contradiction. So $G[S] = K_2$.

Finally, suppose that $e \in E(\overline{G}_i)$; say $i = 1$. Then $G + e$ contains an K-graph H with $e \in E(H)$. If the branch vertices of H are contained in G_2, then $G_1 \cap H$ is a u^1, u^2-path P. Replacing P with $u^1 u^2$ yields a K-graph in $G_2 \subseteq G$, a contradiction. So the branch vertices of H are contained in G_1. If $H \cap G_2 - S \neq \emptyset$ then $H \cap G_2$ is a u^1, u^2 path P. Replacing P by $u^1 u^2$ yields an H graph in $G_1 + e$, proving that G_i is maximum. □

THEOREM 136. *Let $G = (V, E)$ be a graph with $|G| \geq 4$. If G is edge-maximal among graphs with no K-graph then G is 3-connected.*

PROOF. Argue by induction on $|G|$. Suppose $S = \{x, y\}$ is a minimum separating set. If $|S| \geq 3$ we are done, so suppose not. Choose sets V_1 and V_2 such that $V = V_1 \cup V_2$ and $S = V_1 \cap V_2$. By Lemma 135, $G[S] = K_2$ and each $G_i := G[V_i]$ is edge-maximal among graphs with no K-graph. By induction each G_i is either K_3 or 3-connected, so Theorem 134 implies each G_i is planar. For each G_i let $\widetilde{G}_i$ be a plane drawing of G_i. Each face f of $\widetilde{G}_i$ is a triangle, as otherwise we could draw a new edge in f without creating a K-graph. Let xyz_ix be the boundary of a face f_i of G_i. Let v_i be a new vertex and $G_i^+ := G_i \cup K(\{v_i\}, \{x, y, z_i\})$. Drawing v and its incident edges in f_i shows that G_i^+ is planar.

As $e := z_1z_2 \in E(\overline{G})$, $G' := G + e$ contains a K-graph H with $e \in H$. If all the branch vertices of H are contained in the same G_i then there is a path $P = z_iz_{3-i} \dots u$ in G', where $u \in S$. Replacing P by uz_i yields a K-graph in G_i, contradicting Corollary 132. Thus H has branch vertices in each $G_i - S$. In G' there are at most three $V_1 \smallsetminus S, V_2 \smallsetminus S$-paths. Thus the only possibility is that H is a subdivision of $K_{3,3}$ with, say, exactly one branch vertex v in $G_2 - S$. Then $H \cap G'[V_2 + z_1]$ is a $v, \{x, y, z_1\}$-fan F. Thus H with F replaced by $K(v_1, \{x, y, z_1\})$ is a subdivision of $K_{3,3}$ in G_i^+, contradicting Corollary 132. □

PROOF OF KURATOWSKI'S THEOREM 134. First suppose G contains a K-graph. Then by Corollary 132, G is not planar. Now suppose G contains no K-graph. Then G is a spanning subgraph of a graph G' that is edge-maximal among graphs with no K-graph. By Theorem 136, G' is 3-connected. By Theorem 134, G' is planar, so G is planar. □

THEOREM 137 (8.4.32 Thomassen (1994)). *Every planar graph G is 5-list colorable.*

PROOF. It suffices to prove the following more technical statement by induction on G.

CLAIM. Suppose $\widetilde{G}$ is a drawing of a simple planar graph G such that every inner face has length three, and the boundary of the outer face is a cycle $C = v_1v_2 \dots v_sv_1$ with $x = v_1$ and $y = v_2$. If L is a list assignment for G such that

(1) $L(x) = \{\alpha\}$, $L(y) = \{\beta\}$, and $\alpha \neq \beta$,
(2) $|L(v)| = 3$ for all vertices $v \in \{v_3, \dots, v_s\}$, and
(3) $|L(v)| = 5$ for all vertices $v \in V(G - C)$,

then G has an L-coloring.

The claim implies the theorem since adding edges and vertices to G, and deleting colors from some lists of L does not make it easier to L-color G. Moreover, every face of an edge-maximal planar graph is bounded by a C_3. So it suffices to prove the claim.

PROOF OF CLAIM. Argue by induction on $|G|$. Note that $|G| \geq |C| \geq 3$. First consider the base step $|G| = 3$. Color x with α and y with β. The last vertex z has three colors in its list, so it can be colored with a color distinct from α and β.

Now consider the induction step $|G| > 3$.

Case 1: C has a chord v_iv_j with $i > j$. Let $C_1 = v_iv_{i\oplus 1} \dots v_jv_i$ and $C_2 = v_jv_{j\oplus 1} \dots v_iv_j$ be the two nonspanning cycles contained in $C + v_iv_j$. Let $\widetilde{G}_i$ be the plane graph formed by C_i and its interior. Then $\widetilde{G}_1 \cup \widetilde{G}_2 = G$, $\widetilde{G}_1 \cap \widetilde{G}_2 = \widetilde{G}[\{v_iv_j\}]$, and $x, y \in V(C_1)$. By the induction hypothesis, there exists an L-coloring g_1 of $\widetilde{G}_1$. Set $x' = v_i$, $\alpha' = g_1(x')$, $y' = v_j$, $\beta' = g_1(y')$,

$L'(x') = \{\alpha'\}$, $L'(y') = \{\beta'\}$ and $L'(v) = L(v)$ for all vertices of $\widetilde{G}_2 - x' - y'$. Then by the induction hypothesis there exists an L'-coloring g_2 of $\widetilde{G}_2$. It follows that $f = g_1 \cup g_2$ is an L-coloring of G.
Case 2: C does not have a chord. Since every interior face is bounded by a C_3, $G[N(v_s)]$ contains a hamiltonian x, v_{s-1}-path P. Moreover, since C has no chords, the outer face of $\widetilde{G'} = \widetilde{G} - v_s$ is bounded by the cycle $C' = xPv_{s-1}C^*v_1$ $(= x)$. Of course, the interior faces of $\widetilde{G'}$ have length three. Let $\gamma, \delta \in L(v_s)$ be distinct colors not equal to α. Define a list assignment L' for L by

$$L'(v) = \begin{cases} L(v) - \gamma - \delta & \text{if } v \in V(P) - x - v_{s-1} \\ L(v) & \text{else} \end{cases}$$

(and shrinking oversized lists). By induction $\widetilde{G'}$ has an L'-coloring f'. Pick $\varepsilon \in \{\gamma, \delta\}$ with $\varepsilon \neq f'(v_{s-1})$. Finally, extend f' to an L-coloring f of G by setting $f(v_s) = \varepsilon$. □

This completes the proof of the claim and the Theorem. □

HW 61 (*). Let G be a simple planar graph with girth (length of the shortest cycle) k. Prove that $\|G\| \leq \frac{k}{k-2}(|G| - 2)$.

HW 62 (*). Prove that every simple planar graph G with $|G| \geq 4$ has at least four vertices with degree less than six.

HW 63 (+). Prove that every simple planar graph G with $\delta(G) = 5$ has a matching with at most $\frac{1}{5}|G|$ unsaturated vertices.

HW 64 (*). Prove that the vertices of a simple planar graph can be ordered so that every vertex is preceded by at most five of its neighbors. Similarly, prove that the vertices of every planar bipartite graph can be ordered so that each vertex is preceded by at most three of its neighbors.

HW 65 (*). Prove that every planar bipartite graph satisfies $\chi_l(G) \leq 4$. [Hint: Use the previous problem.

HW 66 (+). Prove that every orientation of every X, Y-bigraph has a kernel. [Hint: When is X a kernel?]

HW 67 (*). Prove that every bipartite planar graph G satisfies $\chi_l(G) \leq 3$. [Hint: Use the previous problem.]

CHAPTER 6

Extras

6.1. Lower Bounds on Ramsey's Theorem

THEOREM 138. *For every integer $k \geq 2$ there exists a graph G such that $\omega(G) < k$, $\alpha(G) < k$, and $|G| \geq \lfloor 2^{k/2-1/2} \rfloor$. In other words,* $\mathrm{Ram}(k,k) \geq 2^{k/2-1/2}$.

PROOF. Fix $k \geq 2$, and set $n = \lfloor 2^{k/2-1/2} \rfloor$. Let V be a set of n vertices, and $\mathcal{G}$ be the set of all graphs G with $V(G) = V$. So $G = (V,E) \in \mathcal{G}$ if and only iff $E \subseteq \binom{V}{2}$. Since there are $2^{\binom{n}{2}}$ choices for E,

$$|\mathcal{G}| = N := 2^{\binom{n}{2}}. \tag{6.1.1}$$

For $X \subseteq V$ with $|X| = k$, let $\mathcal{G}_X$ be the set of graphs in $\mathcal{G}$ such that X is a clique or co-clique. So if $G := (V,E) \in \mathcal{G}$ then $G \in \mathcal{G}_X$ iff $E \cap \binom{X}{2} \in \{\emptyset, \binom{X}{2}\}$ and $E \smallsetminus \binom{X}{2} \subseteq \binom{V}{2} \smallsetminus \binom{X}{2}$. There are two possibilities for the first conjunct and $2^{\binom{n}{2}-\binom{k}{2}}$ possibilities for the second. Thus

$$|\mathcal{G}_X| = 2 \cdot 2^{\binom{n}{2}-\binom{k}{2}} = 2 \cdot 2^{-(k^2-k)/2} N. \tag{6.1.2}$$

Any graph G in

$$\mathcal{G} \smallsetminus \bigcup_{X \in \binom{V}{k}} \mathcal{G}_X,$$

satisfies $\omega(G), \alpha(G) < k$ and $|G| = n$. So it suffices to prove $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| < |\mathcal{G}|$. Since (a) $|\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| \leq \binom{n}{k}|\mathcal{G}_X|$, (b) $\binom{n}{k} < \frac{n^k}{k!}$, and (c) $\frac{n}{2^{k/2-1/2}} \leq 1$, this follows from:

$$\begin{aligned}
|\mathcal{G}| - |\bigcup_{X \in \binom{V}{k}} \mathcal{G}_X| &\geq N - \binom{n}{k} \cdot 2 \cdot 2^{-(k^2-k)/2} N && ((a), (6.1.1), (6.1.2)) \\
&> N\left(1 - 2\frac{n^k}{k!} 2^{-(k-1)k/2}\right) && (b) \\
&\geq N\left(1 - \left(\frac{n}{2^{k/2-1/2}}\right)^k\right) \geq 0. && (c)
\end{aligned}$$

□

6.2. Equitable Coloring

DEFINITION 139. An equitable k-coloring of a graph $G = (V,E)$ is a proper coloring $f : V \to [k]$ such that difference $||f^{-1}(i)| - |f^{-1}(j)||$ in the sizes of the the i-th and j-th color classes is at most 1 for all $i, j \in [k]$. In particular, every color is used if $|G| \leq k$.

THEOREM 140 (Hajnal & Szemerédi Theorem (1976)). *Every graph G with maximum degree at most r has an equitable $(r+1)$-coloring.*

The proof was long and sophisticated, and does not provide a polynomial time algorithm. Kierstead and Kostochka found a much simpler and shorter proof. This better understanding has led to many new results, several of which are stated below.

Let $\theta(G) = \max\{d(x) = d(y) : xy \in E(G)\}$.

Theorem 141 (Kiestead & Kostochka (2008)). *For every $r \geq 3$, each graph G with $\theta(G) \leq 2r + 1$ has an equitable $(r + 1)$-coloring.*

Theorem 142 (Kierstead, Kostochka, Mydlarz & Szemerédi). *There is an algorithm that constructs an equitable k-coloring of any graph G with $\Delta(G) + 1 \leq k$, using time $O(r|G|^2)$.*

Problem 143. Find a polynomial time algorithm for constructing the coloring in Theorem (141).

One might hope to prove an equitable version of Brooks' Theorem, but the following example shows that the statement would require special care: For r is odd, $K_{r,r}$ satisfies $\Delta(K_{r,r}) = r$ and $\omega(G) = 2$, but has no r-equitable coloring. Chen, Lih and Wu [?] proposed the following common strengthening of Theorem 140 and Brooks' Theorem.

Conjecture 144. *Let G be a connected graph with $\Delta(G) \leq r$. Then G has no equitable r-coloring if and only if either (a) $G = K_{r+1}$, or (b) $r = 2$ and G is an odd cycle, or (c) r is odd and $G = K_{r,r}$.*

Kierstead and Kostochka have proved the conjecture for $r \leq 4$, and also for $r \geq \frac{1}{4}|G|$.

Proof of Theorem 140. Let G be a graph with $\Delta(G) \leq r$. We may assume that $|G|$ is divisible by $r + 1$: If $|G| = s(r + 1) - p$, where $p \in [r]$ then set $G' := G + K_p$. Then $|G'|$ is divisible by $r + 1$ and $\Delta(G') \leq r$. Moreover, the restriction of any equitable $(r + 1)$-coloring of G' to G is an equitable $(r + 1)$-coloring of G. So we may assume $|G| = (r + 1)s$.

We argue by induction on $\|G\|$. The base step $\|G\| = 0$ is trivial, so consider the induction step. Let u be a non-isolated vertex. By the induction hypothesis, there exists an equitable $(r + 1)$-coloring of $G - E(u)$. We are done unless some color class V contains an edge uv. Since $\Delta(G) \leq r$, some color class W contains no neighbors of u. Moving u to W yields an $(r+1)$-coloring of G with all classes of size s, except for one *small* class $V^- := V - u$ of size $s-1$ and one *large* class $V^+ := W + u$ of size $s+1$. Such a coloring is called *nearly equitable*.

Given a nearly equitable $(r + 1)$-coloring, define an auxiliary digraph $\mathcal{H}$, whose vertices are the color classes, so that UW is a directed edge if and only if some vertex $y \in U$ has no neighbors in W. In this case we say that y *witnesses* UW. Let $\mathcal{A}$ be the set of classes from which V^- can be reached in $\mathcal{H}$, $\mathcal{B}$ be the set of classes not in $\mathcal{A}$ and $\mathcal{B}'$ be the set of classes reachable from V^+ in $\mathcal{H}[\mathcal{B}]$. Set $a := |\mathcal{A}|$, $b := |\mathcal{B}|$, $b' := |\mathcal{B}'|$, $A := \bigcup \mathcal{A}$, $B := \bigcup \mathcal{B}$ and $B' := \bigcup \mathcal{B}'$. Then $r + 1 = a + b$. Since every vertex $y \in B$ has a neighbor in every class of $\mathcal{A}$ and every vertex $z \in B'$ also has a neighbor in every class of $\mathcal{B} - \mathcal{B}'$,

(*) $$d_A(y) \geq a \text{ for all } y \in B \text{ and } d_{A \cup B \setminus B'}(z) \geq a + b - b' \text{ for all } z \in B'.$$

Case 0: $V^+ \in \mathcal{A}$. Then there exists a V^+, V^--path $\mathcal{P} = V_1, \ldots, V_k$ in $\mathcal{H}$. Moving each witness y_j of $V_j V_{j+1}$ to V_{j+1} yields an equitable $(r + 1)$-coloring of G. □

We now argue by a secondary induction on b, whose base step $b = 0$ holds by Case 0. Also $|A| = as - 1$ and $|B| = bs + 1$. Now consider the secondary induction step.

A class $W \in \mathcal{A}$ is *terminal*, if every $U \in \mathcal{A} - W$ can reach V^- in $\mathcal{H} - W$. Let $\mathcal{A}'$ be the set of terminal classes, $a' := |\mathcal{A}'|$ and $A' := \bigcup \mathcal{A}'$. An edge wz is *solo* if $w \in W \in \mathcal{A}'$, $z \in B$ and $N_W(z) = \{w\}$. Ends of solo edges are *solo* vertices and *solo neighbors* of each other.

Order $\mathcal{A}$ as $V^-, X_1, \ldots, X_{a-1}$ so that each X_i has a previous out-neighbor.

Case 1: For some $a - b \leq i \leq a - 1$, class X_i is not terminal. Then some $X_j \in \mathcal{A}'$ cannot reach V^- in $\mathcal{H} - X_i$. So $j > i$ and X_j has no out-neighbors before X_i. In particular, $d^+_{\mathcal{A}}(X_j) < b$. Then for each $w \in X_j$, $d_A(w) \geq a - b$, and so $d_B(w) < 2b$. Let S be the set of solo vertices in X_j, and $D := X_j \setminus S$. If $v \in B - N_B(S)$ then v has no solo neighbor in X_j, and so has at least two neighbors in D. Thus $2b\,|D| > 2\,|B - N_B(S)|$. Using $|S| + |D| = s$ and $r\,|S| \geq |E(S, A)| + |N_B(S)|$,

$$bs + (a-1)\,|S| = b\,|D| + r\,|S| > |B - N_B(S)| + |E(S, A)| + |N_B(S)| > bs + |E(S, A)|\,.$$

Thus $(a-1)\,|S| > |E(S, A)|$, and so there exists $w \in S$ with $d_A(w) \leq a-2$. Thus w witnesses some edge $X_jX \in E(\mathcal{H}[A])$. Since $w \in S$, it has a solo neighbor $y \in B$.

Move w to X and y to X_j. This yields nearly equitable colorings of $G[A+y]$ and $G[B-y]$. Since X_j is terminal, $X + w$ can still reach V^-. Thus by Case 0, $G[A + y]$ has an equitable a-coloring. By (*), $\Delta(G[B - y]) \leq b - 1$. So by the primary induction hypothesis $G[B - y]$ has an equitable b-coloring. After combining these equitable colorings we are done.

Case 2: All the last b classes $X_{a-b}, \ldots, X_{a-1}$ are terminal. Then $a' \geq b$. For $y \in B'$, let $\sigma(y)$ be the number of solo neighbors of y. Similarly to (*),

$$r \geq d(y) \geq a + b - b' + d_{B'}(y) + a' - \sigma(y) \geq r + 1 + d_{B'}(y) + a' - b' - \sigma(y).$$

So $\sigma(y) \geq a' - b' + d_{B'}(y) + 1$. Let I be a maximal independent set with $V^+ \subseteq I \subseteq B'$. Then $\sum_{y \in I}(d_{B'}(y) + 1) \geq |B'| = b's + 1$. Since $a' \geq b$,

$$\sum_{y \in I} \sigma(y) \geq \sum_{y \in I}(a' - b' + d_{B'}(y) + 1) \geq s(a' - b') + b's + 1 > a's = |A'|\,.$$

So some vertex $w \in W \in \mathcal{A}'$ has two solo neighbors y_1 and y_2 in the independent set I.

Since the class Y of y_1 is reachable from V^+, we can equitably b-color $G[B - y_1]$. Let Y' be the new class of y_2. If w witnesses an edge WX of $G[A]$ then we are done by Case 1; otherwise we can move w to some class $U \subseteq B' - y_1$. Replacing w with y_1 in W to get W^* and moving w to U yields a new nearly equitable $(r + 1)$-coloring of G. If $U \in \mathcal{A}$ then we are done by Case 0; otherwise at least $a + 1$ classes, W^*, Z', and all $X \in \mathcal{A}' - W$, can reach V^-. In this case we are done by the secondary induction hypothesis.

APPENDIX A

Exceptional notation

$|G| = |V(G)| = n(G)$, number
$\|G\| = |E(G)| = e(G)$, number
$E(v) = \{vu : vu \in E\}$, edge set
$K(A, B) = (A \cup B, \{ab : a \neq b \wedge (a, b) \in A \times B\})$, graph
$K(A) = K(A, A)$, graph
$E_G(A, B) = \{ab \in E(G) : a \neq b \wedge (a, b) \in A \times B\}$, edge set

APPENDIX B

Standards

B.0.1. MAT 416–Level C.

(1) Theorem 24 (Characterization of bipartite graphs).
(2) Theorem 36 (Euler's Theorem). You may use Lemma 35.
(3) Theorem 46 (Tree Theorem). Provide proofs for required parts of Lemma 44 and Lemma 45.
(4) Theorem 58 (Berge's theorem on maximum matchings).
(5) Corollary 62. You may use Hall's Theorem.
(6) Theorem 68 Petersens's Matching Theorem.
(7) Theorem 76 (Whitney's Theorem).
(8) Lemma 119 (Kernel Lemma).
(9) Euler's Formula Theorem 127 and Theorem 128.

B.0.2. MAT 416–Level A. All MAT 416–Level C and HW #1–3, HW4,5, HW 6 except (+) and:

(1) Theorem 5 (Ramsey's Theorem for graphs).
(2) Corollary 26 (Dirac's Theorem). Provide proofs for required parts of Theorem 25.
(3) Theorem 55 (Max-Flow, Min-Cut Theorem). You may use Theoerm 54.
(4) Theorem 60 (König–Egerváry Theorem) using Max-Flow, Min-Cut Theorem.
(5) Corollary 61 (Hall's Theorem) using König, Egerváry Theorem.
(6) Theorem 65 (Tutte's Theorem).
(7) Theorem 69 (Petersen's 2-Factor Theorem).
(8) Theorem 79 (Thomassen's Contraction Theorem).
(9) Theorems 86, 87, 88, 89, 91, 92, and 94 (Various versions of Menger's Theorem). You only need to be able to prove Theorem 86, but you should know and be able to use the others, especially 89, 91, 92, and 94.
(10) Example 98 Mycielski's Construction.
(11) Theorem 101 (Brooks' Theorem). You may assume Proposition 99 and Lemma 100, but be prepared to prove them if asked.
(12) Theorem 105 (Turan's Theorem). You may assume Lemma 104, but be prepared to prove it if asked.
(13) Theorem 109 (Vizing's Theorem). This includes proving Lemma 108.
(14) Theorem 120 (Galvin's Theorem).
(15) Kuratowski's Theorem 130. You may assume:131, 132 and 133. You should be able to prove Theorems 134 and 136, and Theorem 130, using Theorems 134 and 136 (easy).
(16) Thomassen's 5-Choosability Theorem 137.
(17) Lower bound on $\mathrm{Ram}(k, k)$ Theorem 138.

B.0.3. MAT–513. All MAT 416–Level A, HW #1–3, HW #4,5, HW#6.

B.1. Sample MAT 416/513 Midterm 1

Directions: Use one sheet per problem. Order the sheets by problem before submitting.

MAT **513 Students:** Do the **last 4** problems.

MAT **416 Students:** Do **any 6** problems; one will be treated as extra credit. The first four problems are intended to be somewhat easier. Only the six chosen problems should be turned in.

(1) Prove: If an acyclic graph G satisfies $|G| = \|G\| + 1$ then it is connected.
(2) Prove: Every k-regular bigraph G has a perfect matching.
(3) Let $P \subseteq G$ be an x, y-path. Prove: $G[P]$ contains an x, y-path Q with $Q = G[Q]$.
(4) Let T be a forest (acyclic graph) such that $2k$ of its vertices have odd degree. Prove that T decomposes into k paths.

****

(5) Prove: A graph is bipartite if and only if it contains no odd cycle.
(6) Prove: A graph $G = (V, E)$ has a perfect matching if $o(G - S) \leq |S|$ for all $S \subseteq V$.
(7) Let $n \geq 2$ and $d_1, \dots, d_n \in \mathbb{Z}^+$. Prove: If $\sum_{i=1}^{n} d_i = 2n - 2$ then there is a tree with vertices $v_1, \dots, v_n$ such that $d(v_i) = d_i$ for all $i \in [n]$.
(8) For $k, n \in \mathbb{N}$, let G be an A, B-bigraph with $|A| = n = |B|$ such that $\delta(G) \geq k$, and for all $X \subseteq A, Y \subseteq B$, if $|X|, |Y| \geq k$ then $|E(X, Y)| \neq \emptyset$. Prove: G has a perfect matching.

APPENDIX C

Matching card trick

Consider a deck of $2k+1$ cards numbered $1, \ldots, 2k+1$, and denoted by $[2k+1]$. The class chooses a *hand* H consisting of $k+1$ of these cards, and gives them to Professor A. Professor A looks at them, puts one of them in his pocket, and then has a student spread the remaining k cards face-up on a table. Professor B, who has observed none of this transaction, now enters the room, looks at the cards on the table and identifies the one in Professor A's pocket. How is this done?

Solution. Our arithmetic is done modulo $k+1$, and we use $k+1$ instead of 0 for the representative of its equivalence class. Arrange the cards of H in order as $c_1 < \cdots < c_{k+1}$. Let $x = \sum_{c \in H} c \mod k+1$. Professor A hides card c_x. When Professor B arrives, he sees that the cards $d_1 < \cdots < d_{k+1}$ in $[2k+1] \setminus (H - c_x)$ are missing, and he calculates $y := \sum_{c \in H - x} = x - c_x$. The class is holding $c_x - 1 - (x-1) = -y$ cards less than c_x and Professor A is holding c_x. It follows that $c_x = d_{1-y}$, and Professor B can calculate the rhs.

Another way of saying this is that Professor B knows the missing cards $\overline{d}_1 > \cdots > \overline{d}_{k+1}$. Then $c_x = \overline{d}_{k+2-(1-y)} = \overline{d}_y$.

APPENDIX D

Alternative proofs

D.1. Hall's Theoerm

Proof of Theorem **??**. If M is a matching saturating X and $S \subseteq X$ then $|S| = |E(S,Y) \cap M| \le |N(S)|$; so (3.3.2) holds.

Suppose (3.3.2) holds for some X,Y-bigraph with no matching saturating X; among such counterexamples choose G with $|G|$ minimal, and subject to this $\|G\|$ maximal. By minimality, (1) $N(X) = Y$: if $y \in Y \smallsetminus N(X)$ then $G - y$ is a smaller counterexample. Also (2) all $a \in X$ satisfy $N(a) \neq Y$: else, since $G - a$ has a matching saturating $X - a$ and there is an unsaturated vertex $b \in Y$ by (3.3.2), G has a matching saturating X.

Let $a \in X$; by (2) there is $b \in Y$ with $ab \notin E$. Since $G + ab$ satisfies (3.3.2), maximality implies $G + ab$ has a matching M^+ saturating X. By (1) there is $a' \in X$ with $a'b \in E$; by (2) there is $b' \in Y$ with $a'b' \notin E$. Again by maximality, $G + a'b'$ has a matching L^+ saturating X. Set $M := M^+ - ab$ and $L := L^+ - a'b'$.

Let H be the spanning submultigraph of G with edge set $M \cup L + a'b$, where edges in $M \cap L$ have multiplicity 2. (Figure D.1.1). Then $\Delta(H) \le 2$. As G is bipartite, Proposition 57 implies the components of H are paths and even cycles. Each cycle has a perfect matching. As $d_H(a) = 1$ and $d_H(x) = 2$ for all $x \in X - a$, each path P has an end $y \in Y$ and a matching saturating $V(P) - y$. Combining these matchings yields a matching saturating X.

Here is an alternative way through the last paragraph.

Let H be the spanning subgraph of G with $E(H) = M \triangle L$. (Figure D.1.1). Then $\Delta(H) \le 2$ and $d_H(b), d_H(b') \le 1 = d_H(a) = d_H(a') < d_H(x)$ for all $x \in X \smallsetminus \{a, a'\}$. So the component of H containing a is an alternating a,v- path P. If $v \in Y$ then P is an

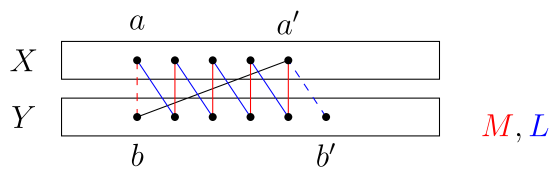

Figure D.1.1. X,Y-bigraph $H \subseteq G$

M-augmenting path. Else $v = a'$, and $aPvb$ is an M-augmenting path. Anyway, G has a matching saturating X. □

D.2. König, Egerváry Theorem

Proof of Theorem ??. Order V as $v_1 \prec \cdots \prec v_{|G|}$. Since W is a cover, every edge is incident to some vertex of W (possibly two). Define a function $g : M \to W$ by $g(e)$ is the least $w \in e \cap W$. Since M is a matching, no vertex of W can be incident to two edges of M. So g is an injection. Thus $|M| \leq |W|$.

Now suppose G is an X, Y-bigraph. Let U be the set of unsaturated vertices in X. Set $m = \{(x, y) \in X \times Y : xy \in M\}$. Since the ends of M in X are distinct, m is a function with domain $X \smallsetminus U$. Since the ends of m in Y are distinct, m is an injection.

If $U = \emptyset$ then X is a cover with $|W| \leq |X| = |M| \leq |W|$; so suppose $U \neq \emptyset$. Letting $A \subseteq V(G)$ be the set of ends of alternating paths starting in U, set $S := A \cap X$, $\overline{S} := X \smallsetminus S$, $T := A \cap Y$, and $\overline{T} := Y \smallsetminus T$. Then $U \subseteq S$ (witnessed by trivial paths). Consider any alternating path $P = v_0 \dots v_n$ with $v_0 \in U$. If i is even then $v_i \in S$, and if also $i \neq 0$ then $v_{i-1}v_i \in M$; if i is odd then $v_i \in T$. We first show:

(D.2.1) $$\text{(i) } N(S) \subseteq T \text{ and (ii) } T \subseteq m(S \smallsetminus U).$$

(i) Let $z \in N(S)$; say $wz \in E(S, z)$. Then there is an alternating path $Q = y_0 \dots y_{2k} (= w)$ with $y_0 \in U$ and $y_{2k-1}w \in M$. Either $z \in V(Q)$ or Qwz is an alternating path starting in U. Regardless, $z \in T$, proving (D.2.1.i).

(ii) Let $z \in T$. Then there is an alternating path $P = y_0 \dots y_{2k+1} (= z)$ with $y_0 \in U$ and $y_{2k}z \notin M$. Since M is maximum, G has no augmenting path. So z is saturated; say $zx \in M$. Then Pzx is an alternating path; so $x \in S \smallsetminus U$, and $z \in m(S \smallsetminus U)$, proving (D.2.1.ii).[1]

The set $W = \overline{S} \cup T$ is a vertex cover of G: Suppose $xy \in E$ with $x \in X$. If $x \in S$ then $y \in T \subseteq W$ by (i); else $x \in \overline{S} \subseteq W$.

It remains to show that $|W| \leq |M|$. For this it suffices to prove that g is onto. Consider $w \in W$. If $w \in T$ then there is $x \in S$ with $xw \in M$ by (ii). As $x \notin W$, $g(xw) = w$. If $w \in \overline{S}$ then there is y with $wy \in M$, since $U \subseteq S$. As M is a matching, (ii) implies $y \in \overline{T}$. So $g(wy) = w$. □

D.3. Menger's Theorem

Theorem 145 (Menger 1927 4.2.17). *Let $G = (V, E)$ be a graph, and suppose $A, B \subseteq V$. Then the size $l := l(A, B)$ of a maximum set of disjoint A, B-paths is equal to the size $k := k(A, B)$ of a minimal A, B-separating set.*

Proof 1. ($l \leq k$) If $\mathcal{P}$ is a set of disjoint A, B-paths and S is an A, B-separator then S must contain at least one vertex of each path, and each vertex of S is on at most one path of $\mathcal{P}$. Thus the function $f : \mathcal{P} \to S$ defined by setting $f(P)$ equal to the first $x \in S \cap V(P)$ is an injection; so $|\mathcal{P}| \leq |S|$. Choosing $\mathcal{P}$ maximum and S minimum yields the inequality.

($k \leq l$) For a set of A, B-paths $\mathcal{P}$ let $\operatorname{end}(\mathcal{P})$ denote the set of ends in B of paths in $\mathcal{P}$. It suffices to show (*) if $\mathcal{P}'$ is a set of disjoint A, B-paths with $|\mathcal{P}'| < k$ then there exists a set $\mathcal{P}$ of disjoint A, B-paths such that $|\mathcal{P}| = |\mathcal{P}'| + 1$ and $\mathit{end}(P') \subseteq \mathit{end}(\mathcal{P})$. Argue by induction

[1](i) and (ii) imply the sufficiency of Hall's Criteria (3.3.2): $|N(S)| \leq |S \smallsetminus U|$, so (3.3.2) fails if $U \neq \emptyset$.

on $|G'|$, where $G' := G - B$. If $|G'| = 0$ then $A \subseteq B$. So the A,B-paths are exactly the paths consisting of a single vertex of A, and $k = |A|$. Thus (*) holds.

Suppose $|G'| > 0$, and fix a set $\mathcal{P}'$ of disjoint A,B-paths with $|\mathcal{P}'| < k$. Since $|\text{end}(\mathcal{P}')| = |\mathcal{P}'| < k$, there is an A,B-path $R = Ry'$ in $G - \text{end}(\mathcal{P}')$. If $R \cap \bigcup \mathcal{P}' = \emptyset$ then put $\mathcal{P} := \mathcal{P}' + R$. If not, let x be the last vertex of R that is in $\bigcup \mathcal{P}'$; say $x \in P \in \mathcal{P}'$ and y is the end of P in B. Note that $x \notin B$, since $y \neq y'$ and $V(P) \cap B = y$. Put $B' := B \cup V(xRy' \cup xPy)$ and $\mathcal{Q}' := \mathcal{P}' - P + Px$. Then $\text{end}(\mathcal{Q}') = \text{end}(\mathcal{P}') - y + x$. Since every A,B-path contains an A,B'-path, $k = k(A,B) \leq k(A,B')$. Since $B + x \subseteq B'$, $|G - B'| < |G - B|$. So by induction, there exists a set $\mathcal{Q}$ of $A,B^{'}$-paths and $y'' \in B'$ such that $|\mathcal{Q}| - 1 = |\mathcal{Q}'| = |\mathcal{P}'|$ and $\text{end}(\mathcal{Q}) = \text{end}(\mathcal{Q}') + y''$. So $x \neq y''$. Let x, y'' be the ends of $Q, Q'' \in \mathcal{Q}$. Set $\mathcal{P}_0 = \mathcal{Q} - Q - Q''$. If $y'' \in xPy$ then set $\mathcal{P} := \mathcal{P}_0 + QxRy' + Q''y''Py'$; if $y'' \in xRy'$ then set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''y''Ry'$; else set $\mathcal{P} := \mathcal{P}_0 + QxPy + Q''$. Evidently, $\mathcal{P}$ witnesses (*). □

PROOF 2 ($k \leq l$). So it suffices to show $k \leq l$. Argue by induction on $\|G\|$.
Base Step: $\|G\| = 0$. Then every A,B-path is trivial. So $A \cap B$ is the maximum set of disjoint A,B-paths and the minimum A,B-separating set. Thus $l = |A \cap B| = k$.
Induction Step: $\|G\| \geq 1$. Let $e = xy \in E(G)$, and put $G' = G \cdot e$. For any $U \subseteq V$, define

$$U' = \begin{cases} U - \{x,y\} + v_e & \text{if } U \cap \{x,y\} \neq \emptyset \\ U & \text{otherwise} \end{cases},$$

and note that for every $T \subseteq V(G')$ there exists $S \subseteq V$ with $T = S'$. Using Lemma 78 and the discussion before, every set $\mathcal{P}'$ of disjoint A',B'-paths corresponds to a set $\mathcal{P}$ of disjoint A,B-paths with $|\mathcal{P}| = |\mathcal{P}'|$ (but not vice versa). So

$$l_{G'}(A',B') \leq l.$$

Also, if S is an A,B-separator in G if and only if S' is an A',B'-separator in G'. So

$$k_{G'}(A',B') \leq k \leq k_{G'}(A',B') + 1.$$

Choose a minimum A',B'-separator T in G'. If $k_{G'}(A',B') = k$ then by the induction hypothesis applied to G' we have:

$$k = k_{G'}(A',B') \leq l_{G'}(A',B') \leq l,$$

and we are done. Otherwise, $k = k_{G'}(A',B') + 1$. In this case $v_{xy} \in T$, and $T = S'$, where $S := T - v_{xy} + x + y$. In particular $xy \in G[S]$.

Set $G'' = G - e$. Since $e \in G[S]$,

$$k_G(A,S) = k_{G''}(A,S) \text{ and } k_G(B,S) = k_{G''}(B,S) \tag{D.3.1}$$

Since S separates A from B in G, every A,S-separator in G separates A from B, and so has size at least $|S|$, and a similar statement holds for B. So we have

$$k_G(A,S), k_G(S,B) \geq k. \tag{D.3.2}$$

Thus

$$|S| \geq l_G(A,S) \geq l_{g''}(A,S) =_{i.h.} k_{G''}(A,S) =_{(\text{D.3.1})} k_G(A,S) \geq_{(\text{D.3.2})} k = |S| \text{ and}$$
$$|S| \geq l_G(B,S) \geq l_{g''}(B,S) =_{i.h.} k_{G''}(B,S) =_{(\text{D.3.1})} k_G(B,S) \geq_{(\text{D.3.2})} k = |S|.$$

Let $\mathcal{K}_A$ be a collection of $|S| = k$ disjoint A, S-paths and $\mathcal{K}_B$ be a collection of $|S|$ disjoint S, B-paths. Then for each $z \in S$ there is a unique A, z-path P_z and a unique z, B-path Q_z. If $v \in V(P_w) \cap V(Q_z)$ then $v \in S$, since otherwise PvQ is an A, B-walk in $G - S$, contradicting the fact that S is an A, B-separator. Thus $w = v = z$, and so $\{P_z z Q_z : z \in S\}$ is a collection of $|S| = k$ disjoint A, B-paths. □

DEFINITION 146 (4.2.18). The line graph $H = L(G)$ of a graph $G = (V, E)$ is defined by

$$V(H) = E \text{ and } E(H) = \{ee' : e \cap e' \neq \emptyset\}.$$

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