WENO-LF

We use a third-order WENO-LF (weighted essentially non-oscillatory Lax-Friedrichs) finite difference method for our supersonic astrophysical flow simulations. We have extended and adapted the code for simulating very high Mach number flows with radiative cooling.

ENO and WENO schemes are high-order finite difference schemes designed for nonlinear hyperbolic conservation laws with piecewise smooth solutions containing sharp discontinuities like shock waves and contacts. Locally smooth stencils are chosen via a nonlinear adaptive algorithm to avoid crossing discontinuities whenever possible in the interpolation procedure. The weighted ENO schemes use a convex combination of all candidate stencils, rather than just one as in the original ENO method.

We now describe the computational procedure for the third-order WENO scheme in more detail. Spatial discretization is discussed first. We start with the simple case of a scalar equation

$$u_t + f(u)_x = 0$$

and assume $\partial f(u)/\partial u \geq 0$, i.e., that the “wind direction” is positive. More general cases will be described later. The computational domain is discretized into a uniform mesh of $N$ gridpoints $x_i = i\Delta x, \ i = 1, 2, \ldots, N$ where $\Delta x$ is the uniform mesh size. A smooth nonuniform mesh could also be used to concentrate gridpoints in certain regions to obtain better resolution. A conservative numerical approximation $u_j(t)$ to the exact solution $u(x_j, t)$ of (1) satisfies the following ODE system:

$$\frac{du_j(t)}{dt} + \frac{1}{\Delta x} \left( \hat{f}_{j+1/2} - \hat{f}_{j-1/2} \right) = 0$$

where $\hat{f}_{j+1/2}$ is called the numerical flux, the design of which is the key ingredient for a successful scheme. For the third-order WENO scheme, the numerical flux $\hat{f}_{j+1/2}$ is defined as follows:

$$\hat{f}_{j+1/2} = \omega_1 \hat{f}^{(1)}_{j+1/2} + \omega_2 \hat{f}^{(2)}_{j+1/2}$$

where $\hat{f}^{(m)}_{j+1/2}$, for $m = 1, 2$, are the two second-order accurate fluxes on two different stencils given by

$$\hat{f}^{(1)}_{j+1/2} = -\frac{1}{2} f_{j-1} + \frac{3}{2} f_j, \ \hat{f}^{(2)}_{j+1/2} = \frac{1}{2} f_j + \frac{1}{2} f_{j+1}.$$
The nonlinear weights $\omega_m$ are given by

$$\omega_m = \frac{\hat{\omega}_m}{\sum_{l=1}^{2} \hat{\omega}_l}, \quad \hat{\omega}_l = \frac{\gamma_l}{(\varepsilon + \beta_l)^2} \quad (5)$$

with the linear weights $\gamma_l$ given by

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3} \quad (6)$$

and the smoothness indicators $\beta_l$ by

$$\beta_1 = (f_j - f_{j-1})^2, \quad \beta_2 = (f_{j+1} - f_j)^2 \quad (7)$$

Finally, the parameter $\varepsilon$ insures that the denominator in Eq. (5) never becomes 0, and is fixed at $\varepsilon = 10^{-6}$ in the computations presented here. The choice of $\varepsilon$ does not affect accuracy: the numerical errors can be much lower than $\varepsilon$, approaching machine zero. Note that we have used the short-hand notation $f_j$ to denote $f(u_j(t))$, and that the stencil for the scheme is biased to the left because of the positive wind direction.

This completes the description of the third-order finite difference WENO scheme for the scalar equation with a positive wind direction. As we can see, the algorithm is actually quite simple and there are no parameters to be tuned in the scheme. The main reason that it works well, both for smooth solutions and for solutions containing shocks or other discontinuities or high gradient regions, is that the nonlinear weights, determined by the smoothness indicators, automatically adjust themselves based on the numerical solution to use the locally smoothest information given by the solution. Higher order WENO schemes are available along the same lines.

If the wind direction $\partial f(u)/\partial u \leq 0$, the method for computing the numerical flux $\hat{f}_{j+1/2}$ is the exact mirror image with respect to the point $x_{j+1/2}$ of the description above. The stencil would then be biased to the right. If $\partial f(u)/\partial u$ changes sign, we use a smooth flux splitting

$$f(u) = f^+(u) + f^-(u) \quad (8)$$

where $\partial f^+(u)/\partial u \geq 0$ and $\partial f^-(u)/\partial u \leq 0$, and apply the above procedure separately on each of them. There are many choices of such flux splittings; the most popular one is the Lax-Friedrichs flux splitting where

$$f^\pm(u) = \frac{1}{2} (f(u) \pm \alpha u) \quad (9)$$
with $\alpha = \max_u |\partial f(u)/\partial u|$.

For hyperbolic systems of conservation laws (1), the eigenvalues of the Jacobian $\partial f(u)/\partial u$ are all real, and there is a complete set of right and left eigenvectors. This allows us to apply the nonlinear WENO procedure in each of the local characteristic fields, obtained by using the left eigenvectors of the Jacobian. For multiple spatial dimensions, the finite difference version of WENO schemes simply applies the WENO procedure in each direction to obtain high order approximations to the relevant spatial derivatives. Unlike dimensional splitting, such a dimension by dimension method allows us to obtain high order accuracy without the computational cost of truly multidimensional reconstructions.

The time discretization is implemented by a third-order TVD Runge-Kutta method:

\[
\begin{align*}
    u^{(1)} &= u^n + \Delta t L(u^n, t^n) \\
    u^{(2)} &= \frac{3}{4} u^n + \frac{1}{4} u^{(1)} + \frac{1}{4} \Delta t L(u^{(1)}, t^n + \Delta t) \\
    u^{n+1} &= \frac{1}{3} u^n + \frac{2}{3} u^{(2)} + \frac{2}{3} \Delta t L(u^{(2)}, t^n + \frac{1}{2} \Delta t)
\end{align*}
\]  

(10)

where $L$ is the approximation of the spatial derivatives $L(u, t) \approx -\partial f(u)/\partial x$ by the WENO procedure outlined above. The time discretization is stable if the first-order forward Euler time-stepping of the spatial operator is stable. This time discretization is very simple and consists of convex combinations of three first-order forward Euler steps. A CFL condition is needed for stability:

\[
\alpha \frac{\Delta t}{\Delta x} \leq CFL
\]  

(11)

where $\alpha$ should be taken as the largest (in absolute value) eigenvalue of the Jacobian $\partial f(u)/\partial u$. The CFL number should be less than one for stability and in our computations it is taken to be between 0.1 and 0.6 (typically 0.4) depending on the stiffness of the cooling source term.