Hyperbolic Conservation Laws

The 1D conservation law

\[ w_t + f(w)_x = 0 = w_t + \left( \frac{\partial f}{\partial w} \right)_x w_x \]

(where \( w \) and \( f \) have \( m \) components) is hyperbolic if the Jacobian matrix \( \frac{\partial f}{\partial w} \) has real eigenvalues and is diagonalizable (complete set of linearly independent eigenvectors). Then solutions to the conservation law can be viewed in terms of propagating waves. If the eigenvalues of \( \frac{\partial f}{\partial w} \) are real and distinct, then the eigenvectors are guaranteed to form a complete linearly independent set, and the conservation law is called strictly hyperbolic.

Finite Volume Methods

Finite volume methods are conservative by construction. Define the cell average of \( w(x, t_n) \) as

\[ w^n_i = \frac{1}{\Delta x} \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} w(x, t_n) \, dx \]

where \( i \pm \frac{1}{2} \) is a shorthand for \( x_{i+\frac{1}{2}} \). Then integrate the conservation law \( w_t + f(w)_x = 0 \) with respect to \( t \) and \( x \) to obtain

\[ \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} w(x, t_{n+1}) \, dx - \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} w(x, t_n) \, dx = -\int_{t_n}^{t_{n+1}} f \left( w \left( x_{i+\frac{1}{2}}, t \right) \right) \, dt + \int_{t_n}^{t_{n+1}} f \left( w \left( x_{i-\frac{1}{2}}, t \right) \right) \, dt \]

and the conservative form

\[ w^{n+1}_i = w^n_i - \frac{\Delta t}{\Delta x} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) \]

where the numerical flux function

\[ F_{i+\frac{1}{2}} \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f \left( w \left( x_{i+\frac{1}{2}}, t \right) \right) \, dt. \]
Solutions to Hyperbolic Conservation Laws

A *discontinuous solution* (or *generalized solution*) to a hyperbolic conservation law can be verified by using the integral formulation

\[
\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left( w_t + f(w)_x \right) = 0
\]

or the weak (often more convenient) formulation

\[
\int_0^\infty dt \int_{-\infty}^{\infty} dx \left( w \phi_t + f(w) \phi_x \right) = -\int_{-\infty}^{\infty} dx \phi(x,0)w(x,0)
\]

of the conservation law, or by showing that the solution satisfies the conservation law piecewise in smooth regions and the Rankine-Hugoniot jump conditions \( s[w] = [f] \) at discontinuities.

A *physical solution* (which is stable to small perturbations) is verified by showing it satisfies an entropy condition or is the vanishing viscosity limit solution.