Numerical Methods

Accuracy in presence of a discontinuity

\[ u_t + a u_x = 0 \]

With LF or upwind, \( u_t + a u_x = D u_{xx} \) where

\[ D_{LF} = \frac{\Delta x^2}{2 \Delta t} (1 - r^2) \]

\[ D_{upwind} = \frac{a \Delta x}{2} (1 - r) \]

\[ r = \frac{a \Delta t}{\Delta x} \]

\[ u_D(x,t) = 1 - \text{erf}\left(\frac{x-at}{\sqrt{4 Dt}}\right) \]

\[ \| u(\cdot,t) - u_D(\cdot,t) \| = 2 \int_{-\infty}^{0} \text{erf}\left(\frac{x}{\sqrt{4Dt}}\right) dx \]

\[ = 2 \sqrt{4Dt} \int_{-\infty}^{0} \text{erf}(x) dx \]

\[ = C_1 \sqrt{Dt} \approx C_2 \sqrt{\Delta x \Delta t} \quad \text{as} \quad \Delta x \to 0 \quad \text{with} \quad r \text{ fixed} \]

\text{Lax-Wendroff Theorem} \quad \text{Sequence of grids indexed by} \quad l = 1, 2, \ldots \quad \text{with} \quad \Delta t_k, \Delta x_k \to 0 \quad \text{as} \quad l \to \infty. \quad \text{Let} \quad U_l \text{ be numerical approximation computed with a consistent} \quad & \text{conservative method on} \ l \text{th grid. If} \ U_l(x,t) \to u(x,t) \quad \text{as} \quad l \to \infty, \quad \text{then} \ U(x,t) \text{ is a weak solution of the conservation law.} \]

\text{A nonconservative method may converge to a function that} \quad \text{is not a weak solution of the original PDE.} \]

\text{For conservative methods, we still need an entropy condition to converge to the correct weak solution.}
**Upwind Method (Courant - Isaacson - Rees)**

Approx characteristics by straight lines for \( u_t + f(u)_x = 0 \)

For scalar \( u_t + f(u)_x = 0 \), if \( f'(u_{j}^n) > 0 \)

\[
   u_{j}^{n+1} = \frac{1}{\Delta x} \left[ (\Delta x - f'(u_{j}^n) \Delta t) u_{j}^n + f'(u_{j}^n) \Delta t u_{j-1}^n \right] 
   = u_{j}^n - \frac{\Delta t}{\Delta x} f'(u_{j}^n) (u_{j}^n - u_{j-1}^n) 
\]

nonconservative!

**Godunov's Method (1959)**

* Solve Riemann problems forward in time

conservative since RPs are exact solutions to the conservation laws

Define a piecewise constant function \( \tilde{u}^n(x,t) \)

\[
   \tilde{u}^n(x,t_n) = u_j^n \quad \text{for} \quad x_{j-\frac{1}{2}} < x < x_{j+\frac{1}{2}} 
\]

We will obtain \( \tilde{u}^n(x,t) \) for \( t_n \leq t \leq t_{n+1} \) from RP solutions
Define

\[ u_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}(x, t^{n+1}) \, dx \text{ cell average} \]

\[ = \frac{1}{\Delta x} \left[ \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}(x, t^n) \, dx + \int_{t^n}^{t^{n+1}} f(\tilde{u}(x_{j-1/2}, t)) \, dt \right. \]

\[ \left. - \int_{t^n}^{t^{n+1}} f(\tilde{u}(x_{j+1/2}, t)) \, dt \right] \text{ finite-volume method} \]

\[ = u_j^n - \Delta t \frac{\Delta x}{\Delta t} \left[ F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n) \right] \text{ conservation form} \]

where \( F(u_j^n, u_{j+1}^n) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(\tilde{u}(x_{j+1/2}, t)) \, dt \)

\[ = f(\tilde{u}^*(u_j^n, u_{j+1}^n)) \quad \tilde{u}^* = \text{RP solution} \]

CFL condition insures that interactions of RPs do not affect \( \tilde{u}^n(x_{j+1/2}, t) \) since \( \max_{j, p} \left| \frac{\Delta t}{\Delta x} x_p(u_j^n) \right| \leq 1 \)

Here \( \tilde{u}^n(x, t) \) would be difficult or impossible to calculate, but all we need is \( \tilde{u}^n(x_{j+1/2}, t) = \text{const} \)