Derivative Approximations

Second-order accurate central difference approx to first derivative

$$\left( \frac{df}{dx} \right)_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x} = f'_i + \frac{\Delta x^2}{6} f''_i + \cdots$$

First-order accurate backward difference approx to first derivative

$$\left( \frac{df}{dx} \right)_i \approx \frac{f_i - f_{i-1}}{\Delta x} = f'_i - \frac{\Delta x}{2} f''_i + \cdots$$

First-order accurate forward difference approx to first derivative

$$\left( \frac{df}{dx} \right)_i \approx \frac{f_{i+1} - f_i}{\Delta x} = f'_i + \frac{\Delta x}{2} f''_i + \cdots$$

Second-order accurate central difference approx to second derivative

$$\left( \frac{d^2 f}{dx^2} \right)_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} = f''_i + \frac{\Delta x^2}{12} f^{(4)}_i + \cdots$$

Note that the central difference second derivative at $i$ is given by differencing the central difference first derivatives at $i \pm \frac{1}{2}$:

$$f''_i \approx \frac{f'_{i+\frac{1}{2}} - f'_{i-\frac{1}{2}}}{\Delta x} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

with $f'_{i+\frac{1}{2}} = (f_{i+1} - f_i)/\Delta x$ and $f'_{i-\frac{1}{2}} = (f_i - f_{i-1})/\Delta x$.

Forward time difference (with this formula, forward and backward Euler are first-order accurate, while TR is second-order)

$$\frac{du}{dt} \approx \frac{u_{n+1} - u_n}{\Delta t} = u' + \frac{u''}{2} \Delta t + \cdots$$

To verify for example the upwind difference (for flow to the right):

$$\left( \frac{df}{dx} \right)_i \approx \frac{f_i - f_{i-1}}{h} = \frac{1}{h} \left( f_i - \left( f_i - h f'_i + \frac{h^2}{2} f''_i - \cdots \right) \right) = f'_i - \frac{h}{2} f''_i + \cdots$$