Conservative Form

A microscopic conservation law in 3D can be written in PDE form as

$$u_t + \nabla \cdot f(u) = 0.$$

Define the exact conserved quantity $Q = \int_V u \, dV$. To derive the macroscopic conservation law, integrate over a volume and use Gauss’ Theorem:

$$\frac{dQ}{dt} = \int_V \frac{\partial u}{\partial t} \, dV = -\int_{\partial V} f \cdot \mathbf{a} = \text{inflow} - \text{outflow}.$$ 

A numerical method for the 1D conservation law

$$u_t + f(u)_x = 0$$

is conservative if it can be written in the form

$$\frac{du_i(t)}{dt} + \frac{1}{\Delta x} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = 0$$

where $F_{i+\frac{1}{2}}$ is the numerical flux (defined by the numerical method) at $x_{i+\frac{1}{2}}$.

**Proof.** Define $Q = \sum_{i=0}^{N} u_i \Delta x$. Then

$$\frac{dQ}{dt} = \sum_{i=0}^{N} \frac{du_i}{dt} \Delta x = -\sum_{i=0}^{N} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = F_{\frac{-1}{2}} - F_{N+\frac{1}{2}} = \text{inflow} - \text{outflow}.$$ 

Alternatively, a one-step method for the 1D conservation law is conservative if it can be written in the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right).$$

**Proof.** Define $Q^n = \sum_{i=0}^{N} u_i^n \Delta x$. Then

$$\frac{Q^{n+1} - Q^n}{\Delta t} = \sum_{i=0}^{N} \frac{u_i^{n+1} - u_i^n}{\Delta t} \Delta x = -\sum_{i=0}^{N} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = F_{\frac{-1}{2}} - F_{N+\frac{1}{2}} = \text{inflow} - \text{outflow}. $$
Parabolic Conservation Laws

For parabolic conservation laws like the nonlinear diffusion equation $u_t = (D(u)u_x)_x$ with $f(u) = -D(u)u_x$, the forward Euler, backward Euler, TR, and TRBDF2 methods are conservative with numerical flux

$$F_{i+\frac{1}{2}} = f_{i+\frac{1}{2}} = -D_{i+\frac{1}{2}}(u_x)_{i+\frac{1}{2}} = -\frac{1}{2} (D(u_i) + D(u_{i+1})) \frac{u_{i+1} - u_i}{\Delta x}$$

where we have set $D_{i+\frac{1}{2}} = \frac{1}{2} (D(u_i) + D(u_{i+1}))$. Or we may define

$$D_{i+\frac{1}{2}} = D\left(\frac{1}{2}(u_i + u_{i+1})\right).$$

Thus the conservative spatial discretization for the nonlinear diffusion equation is

$$\frac{du_i(t)}{dt} = \frac{1}{\Delta x^2} \left( D_{i+\frac{1}{2}}(u_{i+1} - u_i) - D_{i-\frac{1}{2}}(u_i - u_{i-1}) \right).$$

If $D = \text{const}$, we obtain the conservative spatial discretization for the linear diffusion equation $u_t = Du_{xx}$:

$$\frac{du_i(t)}{dt} = D \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}.$$  

Note: Always discretize $(Du)_x$ (in 3D $\nabla \cdot (D\nabla u)$) directly instead of $Du_{xx} + D_xu_x$, which is nonconservative if $D_x \neq 0$.

Hyperbolic Conservation Laws

For hyperbolic conservation laws, the Lax-Friedrichs, Lax-Wendroff, second upwind, ENO/WENO, and Godunov (PPM, CLAWPACK, etc.) methods are conservative, while the original upwind method is not (original upwind is conservative for $u_t + (cu)_x = 0$ only if $c$ does not change sign). Two-step Lax-Wendroff is manifestly in conservation form. For Lax-Friedrichs, the numerical flux for $u_t + f(u)_x = 0$ is

$$F_{i+\frac{1}{2}} = \frac{1}{2} (f(u_i) + f(u_{i+1})) - \frac{\Delta x}{2\Delta t} (u_{i+1} - u_i).$$