Based on Strang’s *Introduction to Applied Mathematics*

**Method of Steepest Descent**

*Find the minimum of* $P(x) = \frac{1}{2}x^TAx - x^Tb$

The minimum of $P$ solves $Ax = b$: \[-\nabla P = 0 = -Ax + b \iff Ax = b\]

where $r$ is the residual. Make a sequence of guesses $x^{(0)}$, $x^{(1)}$, $x^{(2)}$, . . . to $x$, where \[x^{(k+1)} = x^{(k)} - \alpha \nabla P|_{x^{(k)}}.\]

Here $-\nabla P = r$ is the steepest descent direction. Choose $\alpha$ to minimize $P$ in that direction.

**Example:**

\[P(x) = 2x_1^2 - 2x_1x_2 + x_2^2 + 2x_1 - 2x_2\]
\[\partial_1 P = 4x_1 - 2x_2 + 2\]
\[\partial_2 P = -2x_1 + 2x_2 - 2\]

Start at $x^{(0)} = (0, 0)$. Then the steepest decent direction is \[-\nabla P|_{x^{(0)}} = (-2, 2)\]
\[x^{(1)} = x^{(0)} + (-2\alpha, 2\alpha) = (-2\alpha, 2\alpha)\]
\[P \left( x^{(1)} \right) = 20\alpha^2 - 8\alpha, \text{ minimized by } \alpha = 1/5\]
\[x^{(1)} = (-2/5, 2/5).\]

The next steepest descent direction is \[-\nabla P|_{x^{(1)}} = (2/5, 2/5)\]
\[x^{(2)} = x^{(1)} + (2\alpha/5, 2\alpha/5) = (2(\alpha - 1)/5, 2(\alpha + 1)/5)\]
\[P \left( x^{(2)} \right) = (4\alpha^2 - 8\alpha - 20)/25, \text{ minimized by } \alpha = 1\]
\[x^{(2)} = (0, 4/5).\]

Can show that \[x^{(2k)} = (0, 1 - 1/5^k)\]

which never reaches the exact minimum $(0, 1)$ in a finite number of steps.