Classifying General Nonlinear Systems of PDEs

The nonlinear system of PDEs (note that higher-order time derivatives can always be replaced by adding more state variables and equations)

\[ w_t + F(w, w_x, w_{xx}, \ldots) = 0 \]

is classified by expanding \( w \) as a Fourier perturbation about a constant nonlinear solution \( \bar{w} \)

\[ w = \bar{w} + e^{\sigma t + i k x} \delta w \]

where \( \sigma \) is the growth rate and \( k \) is the wavenumber of the perturbation, and then linearizing with respect to \( \delta w \):

\[ -\sigma \delta w = S \delta w \]

where \( S \) is the \textit{symbol} of the linearized PDE system.

The asymptotic eigenvalues \(-\sigma \) of \( S \) as \( k \to \infty \) determine the mathematical type of the various modes of \( w_t + F(w, w_x, w_{xx}, \ldots) = 0 \):

\[ \sigma \sim \begin{cases} 
\pm ik & \text{hyperbolic} \\
-k^2 & \text{parabolic} \\
+k^2 & \text{unstable} \\
\pm ik^2 & \text{Schrödinger} \\
\pm ik^3 & \text{dispersive} \\
-k^4 & \text{fourth-order diffusive} \\
+k^4 & \text{unstable} 
\end{cases} \]

Typically elliptic modes decouple as a sub-problem in \( S \) as in Poisson’s equation \( \nabla^2 \phi = -\rho \) in electro-gas dynamics or the elliptic constraint in the Navier-Stokes equations \( \nabla \cdot \mathbf{u} = 0 \).

To summarize: In the linearized PDE

\[ u_t \pm cu_x = Du_{xx} \pm \beta u_{xxx} - \epsilon u_{xxxx} \]

the coefficients \( D \) (diffusive) and \( \epsilon \) (fourth-order diffusive) must be \( \geq 0 \) for stability (note the minus sign in front of \( \epsilon \) on the RHS of the PDE), while the advective \( c \) and dispersive \( \beta \) coefficients do not play a role in stability.