Van der Pol Oscillator Equations

The triode oscillator equations (1927) are

$$\frac{d^2y}{dt^2} + (y^2 - \epsilon)\frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0$$

Re-express the second-order ODE as two first-order ODEs with $u = y$, $v = dy/dt$:

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = (\epsilon - u^2)v - u$$

with initial conditions $u(0) = 1$, $v(0) = 0$. For $\epsilon = 4$ (take $t_f = 40$), the attractor in phase space is a closed loop, corresponding to an asymptotically periodic orbit.

Shaw Oscillator Equations

In the van der Pol equations, set $u \to v$ and $v \to -u$, and then add the sinusoidal forcing term to the (wrong!) new dv/dt equation to obtain (up to constants) the Shaw oscillator equations:

$$\frac{du}{dt} = 0.7v + 10u(0.1 - v^2)$$

$$\frac{dv}{dt} = -u + 0.25\sin(1.57w)$$

$$\frac{dw}{dt} = 1.$$

With these parameters, the solution has a strange attractor (Shaw 1981) with fractal dimension ≈ 2.6. For initial conditions, take $u(0) = -0.73$, $v(0) = 0$, $w(0) = 0$ (w is time t), with $t_f = 100$. We can make w periodic with period $2\pi/1.57$ by identifying $w + 2m\pi/1.57 \equiv w$, $m = \pm1, \pm2, \ldots$

Lorenz Equations

The Lorenz equations model Rayleigh-Bénard convection (Edward N. Lorenz, "Deterministic Nonperiodic Flow," *J. Atmospheric Sciences* **20** (1963) 130–141):

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(r - z) - y \\ \frac{dz}{dt} &= xy - bz. \end{aligned}$$

For a counter-rotating vortex, $x(t) \sim$ the angular velocity, $y(t \to \infty) \sim T$ at the middle right edge, and $z(t \to \infty) \sim T$ at the bottom. σ is the Prandtl number[1] (10 is appropriate for cold water; too high for air), r is the Rayleigh number[2], and b is the aspect ratio of the vortex cell.

For $\sigma = 10$, $r = 28$, and $b = 8/3$, the solution has a strange attractor with fractal dimension ≈ 2.06. For initial conditions, take $x(0) = 0$, $y(0) = 1$, and $z(0) = 0$, with $t_f = 30$.

Write the Lorenz equations as

$$\frac{du}{dt} = \frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = f(u) = \begin{bmatrix} \sigma(y - x) \\ x(r - z) - y \\ xy - bz \end{bmatrix}.$$

There are three equilibria $f(u_{eq}) = 0$:

$$(x, y, z)_{eq} = (\eta, \eta, r - 1), \quad (-\eta, -\eta, r - 1), \quad (0, 0, 0)$$

where $\eta = \sqrt{b(r - 1)}$. The eigenvalues λ_i ($i = 1, 2, 3$) of the Jacobian

$$J = \frac{\partial f}{\partial u} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}$$

[1]Dimensionless ratio ν/κ of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

[2]Dimensionless number describing heat flow—if the Rayleigh number is below (above) a critical value, heat transfer is dominated by conduction (convection).

evaluated at $J(u_{eq})$ determine the stability of the equilibria. All three equilibria are unstable for the canonical values $\sigma = 10$, $r = 28$, and $b = 8/3$, with the centers of the "butterfly wings" $(\eta, \eta, r-1)$ and $(-\eta, -\eta, r-1)$ slightly unstable and the origin $(0, 0, 0)$ extremely unstable.

Stability of Equilibria

The equilibria u_{eq} of an IVP system $du/dt = f(u)$ are given by $f(u_{eq}) = 0$. To analyze the stability of the equilibria, make a small perturbation: $u(t) = u_{eq}(t) + \delta u(t)$. Linearizing with respect to δu, we obtain

$$\frac{d\delta u}{dt} = f(u_{eq} + \delta u) \approx J\delta u, \quad J = \left.\frac{\partial f}{\partial u}\right|_{u_{eq}}.$$

Then set $\delta u(t) = e^{\lambda t}\tilde{u}$ where $\tilde{u}$ is a constant. We obtain an eigenvalue problem

$$\lambda \delta u = J\delta u.$$

If $\mathrm{Re}\{\lambda_i\} < 0$ for all i, then the equilibrium is stable. If $\mathrm{Re}\{\lambda_i\} > 0$ for any i, then the equilibrium is unstable.