Van der Pol Oscillator Equations

$$\frac{d^2y}{dt^2} + (y^2 - \epsilon)\frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0$$

Re-express the second-order ODE as two first-order ODEs with $u = y$, $v = dy/dt$:

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = (\epsilon - u^2)v - u$$

with initial conditions $u(0) = 1$, $v(0) = 0$. Take for example $\epsilon = 4$ and $t_f = 40$.

Shaw Oscillator Equations

In the van der Pol equations, set $u \to v$ and $v \to -u$, and then add the sinusoidal forcing term to the (wrong!) new dv/dt equation to obtain (up to constants) the Shaw oscillator equations:

$$\frac{du}{dt} = 0.7v + 10u(0.1 - v^2)$$

$$\frac{dv}{dt} = -u + 0.25\sin(1.57w)$$

$$\frac{dw}{dt} = 1.$$

With these parameters, the solution has a strange attractor (Shaw 1981) with fractal dimension ≈ 2.6. For initial conditions, take $u(0) = -0.73$, $v(0) = 0$, $w(0) = 0$ (w is time t), with $t_f = 100$. We can make w periodic with period $2\pi/1.57$ by identifying $w + 2m\pi/1.57 \equiv w$, $m = \pm 1, \pm 2, \ldots$

Lorenz Equations

The Lorenz equations model Rayleigh-Bénard convection (Edward N. Lorenz, "Deterministic Nonperiodic Flow," *J. Atmospheric Sciences* **20** (1963) 130–141):

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(r - z) - y$$

$$\frac{dz}{dt} = xy - bz.$$

For a counter-rotating vortex, $x(t) \sim$ the angular velocity, $y(t \to \infty) \sim T$ at the middle right edge, and $z(t \to \infty) \sim T$ at the bottom. σ is the Prandtl number[1] (10 is appropriate for cold water; too high for air), r is the Rayleigh number[2], and b is the aspect ratio of the vortex cell.

For $\sigma = 10$, $r = 28$, and $b = 8/3$, the solution has a strange attractor with fractal dimension ≈ 2.06. For initial conditions, take $x(0) = 0$, $y(0) = 1$, and $z(0) = 0$, with $t_f = 30$.

[1]Dimensionless ratio ν/κ of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

[2]Dimensionless number describing heat flow—if the Rayleigh number is below (above) a critical value, heat transfer is dominated by conduction (convection).