Van der Pol Oscillator Equations

\[ \frac{du}{dt} = v \]

\[ \frac{dv}{dt} = (\epsilon - u^2)v - u \]

with initial conditions \( u(0) = 1, v(0) = 0 \). Take for example \( \epsilon = 4 \) and \( t_f = 40 \).

Shaw Oscillator Equations

In the van der Pol equations, set \( u \rightarrow v \) and \( v \rightarrow -u \), and then add the sinusoidal forcing term to the (wrong!) new \( dv/dt \) equation to obtain (up to constants) the Shaw oscillator equations:

\[ \frac{du}{dt} = 0.7v + 10u(0.1 - v^2) \]

\[ \frac{dv}{dt} = -u + 0.25 \sin(1.57w) \]

\[ \frac{dw}{dt} = 1 \]

with initial conditions \( u(0) = -0.73, v(0) = 0, w(0) = 0 \) (\( w \) is time \( t \)). Take \( t_f = 100 \). Note: We can make \( w \) periodic with period \( 2\pi/1.57 \) by identifying \( w + 2m\pi/1.57 \equiv w \), \( m = \pm1, \pm2, \ldots \)

Lorenz Equations

\[ \frac{dx}{dt} = \sigma(y - x) \]

\[ \frac{dy}{dt} = x(r - z) - y \]

\[ \frac{dz}{dt} = xy - bz \]

The Lorenz equations model Rayleigh-Bénard convection (Edward N. Lorenz, “Deterministic Nonperiodic Flow,” J. Atmospheric Sciences 20 (1963) 130–141). For a counter-rotating vortex, \( x(t) \sim \) the angular velocity, \( y(t \rightarrow \infty) \sim T \) at the middle right edge, and \( z(t \rightarrow \infty) \sim T \) at the bottom. \( \sigma \) is the Prandtl number\(^1\) (10 is appropriate for cold water; too high for air), \( r \) is the Rayleigh number\(^2\), and \( b \) is the aspect ratio of the vortex cell.

\(^1\)Dimensionless ratio \( \nu/\kappa \) of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

\(^2\)Dimensionless number describing heat flow—if the Rayleigh number is below (above) a critical value, heat transfer is dominated by conduction (convection).
For $\sigma = 10$, $r = 28$, and $b = 8/3$, the solution is a strange attractor with fractal dimension $\approx 2.06$. Take the initial conditions $x(0) = 0$, $y(0) = 1$, and $z(0) = 0$, with $t_f = 30$. 