Van der Pol Oscillator Equations

The triode oscillator equations (1927) are

\[
\frac{d^2y}{dt^2} + (y^2 - \epsilon)\frac{dy}{dt} + y = 0, \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 0
\]

Re-express the second-order ODE as two first-order ODEs with \( u = y, \ v = \frac{dy}{dt} \):

\[
\frac{du}{dt} = v  \\
\frac{dv}{dt} = (\epsilon - u^2)v - u
\]

with initial conditions \( u(0) = 1, \ v(0) = 0 \). For \( \epsilon = 4 \) (take \( t_f = 40 \)), the attractor in phase space is a closed loop, corresponding to an asymptotically periodic orbit.

Shaw Oscillator Equations

In the van der Pol equations, set \( u \rightarrow v \) and \( v \rightarrow -u \), and then add the sinusoidal forcing term to the (wrong!) new \( \frac{dv}{dt} \) equation to obtain (up to constants) the Shaw oscillator equations:

\[
\frac{du}{dt} = 0.7v + 10u(0.1 - v^2)  \\
\frac{dv}{dt} = -u + 0.25\sin(1.57w)  \\
\frac{dw}{dt} = 1.
\]

With these parameters, the solution has a strange attractor (Shaw 1981) with fractal dimension \( \approx 2.6 \). For initial conditions, take \( u(0) = -0.73, \ v(0) = 0, \ w(0) = 0 \) (\( w \) is time \( t \)), with \( t_f = 100 \). We can make \( w \) periodic with period \( 2\pi/1.57 \) by identifying \( w + 2m\pi/1.57 \equiv w, \ m = \pm 1, \pm 2, \ldots \)
Lorenz Equations


\[
\begin{align*}
\frac{dx}{dt} &= \sigma (y - x) \\
\frac{dy}{dt} &= x(r - z) - y \\
\frac{dz}{dt} &= xy - bz.
\end{align*}
\]

For a counter-rotating vortex, \( x(t) \sim \) the angular velocity, \( y(t \to \infty) \sim T \) at the middle right edge, and \( z(t \to \infty) \sim T \) at the bottom. \( \sigma \) is the Prandtl number\(^1\) (10 is appropriate for cold water; too high for air), \( r \) is the Rayleigh number\(^2\), and \( b \) is the aspect ratio of the vortex cell.

For \( \sigma = 10, r = 28, \) and \( b = 8/3, \) the solution has a strange attractor with fractal dimension \( \approx 2.06. \) For initial conditions, take \( x(0) = 0, y(0) = 1, \) and \( z(0) = 0, \) with \( t_f = 30. \)

Write the Lorenz equations as

\[
\frac{du}{dt} = d\begin{bmatrix} x \\ y \\ z \end{bmatrix} = f(u) = \begin{bmatrix} \sigma(y - x) \\ x(r - z) - y \\ xy - bz \end{bmatrix}.
\]

There are three equilibria \( f(u_{eq}) = 0: \)

\((x, y, z)_{eq} = (\eta, \eta, r - 1), \ (-\eta, -\eta, r - 1), \ (0, 0, 0)\)

where \( \eta = \sqrt{b(r - 1)}. \) The eigenvalues \( \lambda_i \) \( (i = 1, 2, 3) \) of the Jacobian

\[
J = \frac{\partial f}{\partial u} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}
\]

\(^1\)Dimensionless ratio \( \nu/\kappa \) of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

\(^2\)Dimensionless number describing heat flow—if the Rayleigh number is below (above) a critical value, heat transfer is dominated by conduction (convection).
evaluated at $J(u_{eq})$ determine the stability of the equilibria. All three equilibria are unstable for the canonical values $\sigma = 10$, $r = 28$, and $b = 8/3$, with the centers of the “butterfly wings” $(\eta, \eta, r - 1)$ and $(-\eta, -\eta, r - 1)$ slightly unstable and the origin $(0, 0, 0)$ extremely unstable.

**Stability of Equilibria**

The equilibria $u_{eq}$ of an IVP system $du/dt = f(u)$ are given by $f(u_{eq}) = 0$. To analyze the stability of the equilibria, make a small perturbation: $u(t) = u_{eq} + \delta u(t)$. Linearizing with respect to $\delta u$, we obtain

$$\frac{d\delta u}{dt} = f(u_{eq} + \delta u) \approx J\delta u, \quad J = \left. \frac{\partial f}{\partial u} \right|_{u_{eq}}.$$  

Then set $\delta u(t) = e^{\lambda t} \tilde{u}$ where $\tilde{u}$ is a constant. We obtain an eigenvalue problem

$$\lambda \delta u = J\delta u.$$

If $\text{Re}\{\lambda_i\} < 0$ for all $i$, then the equilibrium is stable. If $\text{Re}\{\lambda_i\} > 0$ for any $i$, then the equilibrium is unstable.