Two-Step Lax-Wendroff

For nonlinear hyperbolic conservation laws \( w_t + f(w)_x = 0 \), the two-step Lax-Wendroff (LW) method should be used. It is derived in a manifestly conservative form as a finite volume method using the 2D Gauss Divergence Theorem in \((t, x)\) to discretize \( w_t + f(w)_x = 0 \). Write \( \nabla = (\partial_t, \partial_x) \).

Consider first the spacetime rectangle \( V \) with
\[
x_i \leq x \leq x_{i+1}, \quad t_n \leq t \leq t_{n+\frac{1}{2}}.
\]

Integrating over this rectangle and using the 2D Gauss Divergence Theorem implies
\[
0 = \int_V (w_t + f_x) \, dt \, dx = \int_V \nabla \cdot (w, f) \, dt \, dx = \int_{\partial V} (w, f) \cdot \hat{n} \, ds.
\]

Approximating the integral around the perimeter \( \partial V \) of the rectangle yields
\[
w_{i+\frac{1}{2}}^{n+\frac{1}{2}} \Delta x + f_{i+1}^n \frac{\Delta t}{2} - \frac{1}{2} \left( w_i^n + w_{i+1}^n \right) \Delta x - \frac{\Delta t}{2} f_i^n = 0.
\]

Dividing by \( \Delta x \) and rearranging gives the first step of two-step LW:
\[
w_{i+\frac{1}{2}}^{n+1} = \frac{1}{2} \left( w_i^n + w_{i+1}^n \right) - \frac{\Delta t}{2 \Delta x} \left( f_{i+1}^n - f_i^n \right).
\]

Shifting \( i \to i - 1 \) yields
\[
w_{i-\frac{1}{2}}^{n+1} = \frac{1}{2} \left( w_{i-1}^n + w_i^n \right) - \frac{\Delta t}{2 \Delta x} \left( f_i^n - f_{i-1}^n \right).
\]

Thus these intermediate solutions at \( n + \frac{1}{2}, \ i \pm \frac{1}{2} \) are calculated from the Lax-Friedrichs method.

Next consider the spacetime rectangle \( \tilde{V} \) with
\[
x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}}, \quad t_n \leq t \leq t_{n+1}.
\]

Integrating over this rectangle and using the 2D Gauss Divergence Theorem implies
\[
0 = \int_{\tilde{V}} (w_t + f_x) \, dt \, dx = \int_{\tilde{V}} \nabla \cdot (w, f) \, dt \, dx = \int_{\partial \tilde{V}} (w, f) \cdot \hat{n} \, ds.
\]
Approximating the integral around the perimeter $\partial \tilde{V}$ of the rectangle yields
\[ w_i^{n+1} \Delta x + f_{i+\frac{1}{2}}^{n+\frac{1}{2}} \Delta t - w_i^n \Delta x - \Delta t f_{i-\frac{1}{2}}^{n+\frac{1}{2}} = 0. \]

Dividing by $\Delta x$ and rearranging gives the second step of two-step LW:
\[ w_i^{n+1} = w_i^n - \Delta t \frac{f_{i+\frac{1}{2}}^{n+\frac{1}{2}} - f_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}. \]

Thus the new solution at $n+1$, $i$ is calculated from the leapfrog method.

Two-step LW is second-order accurate, stable for $\Delta t \leq \Delta x / |\lambda|_{\text{max}}$ where $|\lambda|_{\text{max}}$ is the maximum characteristic speed in magnitude, and conservative since its derivation is manifestly conservative.