Two-Step Lax-Wendroff

For nonlinear hyperbolic conservation laws $w_t + f(w)_x = 0$, the two-step Lax-Wendroff (LW) method should be used. It is derived in a manifestly conservative form as a finite volume method using the 2D Gauss Theorem in $(t, x)$ to discretize $w_t + f(w)_x = 0$. Write $\nabla = (\partial_t, \partial_x)$.

Consider first the spacetime rectangle $V$ with $x_i \leq x \leq x_{i+1}, \ t_n \leq t \leq t_{n+\frac{1}{2}}$.

Integrating over this rectangle and using the 2D Gauss Theorem implies

$$0 = \int_{V} (w_t + f_x) \ dt \ dx = \int_{V} \nabla \cdot (w, f) \ dt \ dx = \int_{\partial V} (w, f) \cdot \hat{n} \ ds.$$

Approximating the integral around the perimeter $\partial V$ of the rectangle yields

$$w_{i+\frac{1}{2}}^{n+\frac{1}{2}} \Delta x + f_{i+1}^{n} \frac{\Delta t}{2} - \frac{1}{2} \left( w_{i}^{n} + w_{i+1}^{n} \right) \Delta x - \frac{\Delta t}{2} f_{i}^{n} = 0.$$

Dividing by $\Delta x$ and rearranging gives the first step of two-step LW:

$$w_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left( w_{i}^{n} + w_{i+1}^{n} \right) - \frac{\Delta t}{2\Delta x} \left( f_{i+1}^{n} - f_{i}^{n} \right).$$

Shifting $i \to i - 1$ yields

$$w_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left( w_{i-1}^{n} + w_{i}^{n} \right) - \frac{\Delta t}{2\Delta x} \left( f_{i}^{n} - f_{i-1}^{n} \right).$$

Thus these intermediate solutions at $n + \frac{1}{2}, \ i \pm \frac{1}{2}$ are calculated from the Lax-Friedrichs method.

Next consider the spacetime rectangle $\tilde{V}$ with $x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}}, \ t_n \leq t \leq t_{n+1}$.

Integrating over this rectangle and using the 2D Gauss Theorem implies

$$0 = \int_{\tilde{V}} (w_t + f_x) \ dt \ dx = \int_{\tilde{V}} \nabla \cdot (w, f) \ dt \ dx = \int_{\partial \tilde{V}} (w, f) \cdot \hat{n} \ ds.$$

Approximating the integral around the perimeter $\partial \tilde{V}$ of the rectangle yields

$$w_{i}^{n+1} \Delta x + f_{i+\frac{1}{2}}^{n+\frac{1}{2}} \Delta t - w_{i}^{n} \Delta x - \Delta t f_{i-\frac{1}{2}}^{n+\frac{1}{2}} = 0.$$
Dividing by $\Delta x$ and rearranging gives the second step of two-step LW:

$$w_{i}^{n+1} = w_{i}^{n} - \frac{\Delta t}{\Delta x} \left( f_{i+\frac{1}{2}}^{n+\frac{1}{2}} - f_{i-\frac{1}{2}}^{n+\frac{1}{2}} \right).$$

Thus the new solution at $n+1$, $i$ is calculated from the leapfrog method.

Two-step LW is second-order accurate, stable for $\Delta t \leq \Delta x/|\lambda|_{max}$ where $|\lambda|_{max}$ is the maximum characteristic speed in magnitude, and conservative since its derivation is manifestly conservative.