**Lax–Wendroff Theorem**  \[ u_t + f(u)_x = 0 \]

- **Hope** to approx discontinuous weak solutions to a conservation law by using a conservative method.

LW proved we can do this: if a sequence of computations converge to \( u(x,t) \) as \( \Delta t \to 0, \Delta x \to 0 \), then \( u(x,t) \) will be a weak solution of the conservation law.

- **Thm** does not guarantee convergence — for that we need some form of **stability**.

- In practice, if computed solution looks "reasonable" with well-resolved discontinuities, we can believe it is a good approx to **some** weak solution.

**LW Thm.** Consider a sequence of grids indexed by \( l = 1, 2, \ldots \) with \( \Delta t_1, \Delta x_1 \to 0 \) as \( l \to \infty \). Let \( u_l(x,t) = \) numerical approx computed with a consistent conservative method on the \( l \)th grid. If \( u_l(x,t) \) converges to \( u(x,t) \) as \( l \to \infty \), then \( u(x,t) \) is a weak solution of the conservation law.

**Convergence**

(1) **Over every bounded set** \( \Omega = [a,b] \times [0,T] \) in \( x \& t \),
\[
\| u_l - u \|_{1, \Omega} = \int_0^T \int_a^b \left| u_l(x,t) - u(x,t) \right| \, dx \, dt \to 0 \quad \text{as} \quad l \to \infty
\]
(2) For each $T$ there is an $R > 0$ such that

$$TV(u_x(\cdot, t)) < R \text{ for all } t \in [0, T] \text{ & } x = 1, 2, \ldots$$

$$TV(v) = \sup \sum_{j=1}^{\infty} |v(x_j) - v(x_{j-1})|$$

$$TV(v) = \int_{-\infty}^{\infty} |v'(x)| \, dx$$

$\sup$ is over all partitions of $(-\infty, \infty)$:

$$-\infty = x_0 < x_1 < \ldots < x_{N-1} < x_N = \infty$$

may have $8(x)$'s

LW assumed that $u_x$ converges to $u$ almost everywhere in a uniformly bounded way. Using the fact that each $u_x$ is a piecewise constant function, this assumption is equivalent to (1) & (2).

To prove the LW Thm, we will show that $u(x, t)$ satisfies the weak form of the conservation law:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt \, dx \left( \phi_t u + \phi_x f \right) = -\int_{-\infty}^{\infty} dx \left. \phi(x, 0) u(x, 0) \right|_{-\infty}^{\infty}$$

for all $\phi \in C^1_0$

Weak formulation followed from integrating by parts

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt \, dx \left( \phi u_t + \phi f(u)_x \right) = 0$$
Proof of LW Thm (drop index "\( l \))

Multiply discrete conservation law by \( \phi(x_j, t_n) \) & "sum by parts"

\[
\phi(x_j, t_n) u_{j}^{n+1} = \phi(x_j, t_n) u_{j}^{n} - \frac{\Delta t}{\Delta x} \phi(x_j, t_n) (F_{j+\frac{1}{2}}^{n} - F_{j-\frac{1}{2}}^{n})
\]

\[
\sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi(x_j, t_n) (u_{j}^{n+1} - u_{j}^{n}) = \frac{\Delta t}{\Delta x} \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \phi(x_j, t_n) (F_{j+\frac{1}{2}}^{n} - F_{j-\frac{1}{2}}^{n})
\]

Using summation by parts on \( n \) & \( j \), we get

\[
- \sum_{j=-\infty}^{\infty} \phi(x_j, t_0) u_{j}^{0} - \sum_{n=1}^{\infty} (\phi(x_j, t_n) - \phi(x_j, t_{n-1})) u_{j}^{n} - \frac{\Delta t}{\Delta x} \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} (\phi(x_{j+1}, t_n) - \phi(x_j, t_n)) F_{j+\frac{1}{2}}^{n} = 0
\]

\( \phi \) has compact support, so \( \phi(x_j, t_n) = 0 \) for \(|j| \) or

\( n \) sufficiently large & boundary terms at \( j = \pm \infty, n = \infty \)

drop out

Each sum above is finite since \( \phi \) has compact support

\[
\alpha \Delta t \sum_{n=1}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_j, t_n) - \phi(x_j, t_{n-1})}{\Delta t} u_{j}^{n} +
\]

\[
\sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} \frac{\phi(x_{j+1}, t_n) - \phi(x_j, t_n)}{\Delta x} F_{j+\frac{1}{2}}^{n} = -\Delta x \sum_{j=-\infty}^{\infty} \phi(x_j, t_0) u_{j}^{0}
\]

Now let \( l \to \infty \) so that \( \Delta t \Delta x \to 0 \)
First term in (*) $\rightarrow \int_0^\infty \int_{-\infty}^\infty dt \int dx \Phi_t u$

using $u_t \rightarrow u$ in the 1-norm & smoothness of $\phi$

Third term in (*) $\rightarrow \int_{-\infty}^\infty dx \phi(x,0) u(x,0)$

taking $u_0^x = u(x_j, t_0=0)$

Second term in (*) is more difficult

$F_{j+\frac{1}{2}} = F(u_\ell(x_j-\ell h, t_n), \ldots, u_\ell(x_j+\ell h, t_n))$

for $\ell+\ell+1$ values of $u_\ell$

\[ \left| F(u_\ell(x_j-\ell h, t_n), \ldots, u_\ell(x_j+\ell h, t_n)) - f(u_\ell(x_j, t_n)) \right| \leq K \max_{-\ell \leq k \leq \ell} \left| u_\ell(x_j+kh, t_n) - u_\ell(x_j, t_n) \right| \text{ Lipschitz} \]

$\rightarrow 0$ as $l \rightarrow \infty$ since $u_\ell(\cdot, t)$ has bounded total variation, uniformly in $\ell$, for almost all values of $x$

Therefore $F \approx f$ & second term in (*) $\rightarrow$

$\int_0^\infty \int_{-\infty}^\infty dt \int dx \phi_t f(u) \neq 0$