Fourier Series

The (complex) Fourier series for \( f(x) \) defined for \( 0 \leq x \leq 2\pi \) is the periodic function

\[
f_F(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}
\]

where

\[
a_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-inx} f(x) \, dx
\]

using the orthogonality relation \((m \text{ and } n \text{ are integers})\)

\[
\int_{0}^{2\pi} e^{-inx} e^{inx} \, dx = 2\pi \delta_{mn}.
\]

**Theorem.** If \( f(x) \) is piecewise continuous and has bounded total variation, then

\[
f_F(x) = \frac{1}{2} [f(x_+) + f(x_-)]
\]

for \( 0 \leq x \leq 2\pi \) and \( f_F(x) \) is repeated periodically outside of this interval. Note that

\[
f_F(0) = f_F(2\pi) = \frac{1}{2} [f(0) + f(2\pi)].
\]

For the Fourier sine and cosine series, we make use of the orthogonality relations \((m \text{ and } n \text{ are positive (non-negative) integers for sin (cos)})\)

\[
\int_{0}^{\pi} \sin(mx) \sin(nx) \, dx = \frac{\pi}{2} \delta_{mn}
\]

\[
\int_{0}^{\pi} \sin(mx) \cos(nx) \, dx = 0
\]

\[
\int_{0}^{\pi} \cos(mx) \cos(nx) \, dx = \frac{\pi c_n}{2} \delta_{mn}
\]

where \( c_0 = 2 \) and \( c_n = 1, \, n \geq 1. \)

The Fourier sine series for \( f(x) \) defined for \( 0 < x < \pi \) is the function

\[
f_s(x) = \sum_{n=1}^{\infty} a_n \sin(nx)
\]
where
\[ a_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)f(x) \, dx. \]

The Fourier cosine series is
\[ f_c(x) = \sum_{n=0}^{\infty} a_n \cos(nx) \]

where
\[ a_n = \frac{2}{\pi c_n} \int_0^{\pi} \cos(nx)f(x) \, dx. \]

**Theorem.** If \( f(x) \) is piecewise continuous and has bounded total variation, then
\[ f_s(x) = f_c(x) = \frac{1}{2} \left[ f(x_+) + f(x_-) \right] \]
for \( 0 < x < \pi \). \( f_s(x) \) and \( f_c(x) \) are extended by odd/even symmetry and periodicity outside this interval.

*In the neighborhood of a discontinuity, the Fourier series exhibits the Gibbs phenomenon—a 9% overshoot or undershoot.*

**Rate of convergence of Fourier series.** If \( f(x) \) is smooth and periodic, its Fourier series does not exhibit the Gibbs phenomenon, and the Fourier series converges rapidly and uniformly. If \( f(x) \) is periodic and has continuous derivatives of order \( 0, 1, \ldots, p-1 \) and \( f^{(p)}(x) \) is integrable, then (integrating by parts)
\[ a_n = \frac{1}{2\pi(i)^p} \int_0^{2\pi} e^{-inx} f^{(p)}(x) \, dx \]
alongside \( |a_n| \ll 1/n^p \) as \( n \to \pm\infty \). If \( f(x) \) is infinitely differentiable and periodic, then \( f^{(p)}(x) \) converges to \( f(x) \) more rapidly than any finite power of \( 1/n \) as \( n \to \pm\infty \) for all \( x \) (“infinite” order accuracy). The Fourier sine and cosine series have similar convergence properties.

The Fourier expansion for the square wave on \([-\pi, \pi]\) is
\[ \theta_s(x) = \frac{4}{\pi} \left[ \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots \right]. \]

Note that \( \theta_s(\pi) = 0 \). The Fourier approximation to the square wave exhibits the Gibbs phenomenon: the Gibbs overshoot is \( \theta_s(\pi_-) \approx 1.17898 \) even in the limit of an infinite number of Fourier modes.

2
Chebyshev Polynomials

The Chebyshev polynomials $T_n(x)$, $-1 \leq x \leq 1$, are defined by $T_n(\cos \theta) = \cos(n\theta)$. The first few Chebyshev polynomials are $T_0 = 1$, $T_1 = x$, $T_2 = 2x^2 - 1$, $T_3 = 4x^3 - 3x$, $T_4 = 8x^4 - 8x^2 + 1$. The extrema of $T_N(x)$ are $x_j = \cos(\pi j/N)$, $j = 0, \ldots, N$. Note that $T_n(x)$ is even (odd) if $n$ is even (odd) and that $T_n(\pm1) = (\pm1)^n$.

Figure 1: Chebyshev polynomials $T_1$, $T_2$, $T_4$, and $T_8$.

The Chebyshev polynomials obey the orthogonality relation

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} T_m(x)T_n(x) = c_n \frac{\pi}{2} \delta_{mn}$$

where $c_0 = 2$ and $c_n = 1$, $n \geq 1$, and the recursion formula

$$\frac{T'_n+1}{n+1} - \frac{T'_n}{n-1} = \frac{2}{c_n} T_n$$

with $T'_{-1} \to 0$ and $T'_0 \to 0$.

For $y(x) = \sum_{n=0}^{N} a_n T_n(x)$, the Chebyshev first derivative formula is $y'(x) = \sum_{n=0}^{N-1} b_n T'_n(x)$ with

$$c_n b_n = 2 \sum_{p=n+1}^{N} p a_p, \quad (p + n \text{ odd})$$
and the Chebyshev second derivative formula is $y''(x) = \sum_{n=0}^{N-2} b_n^{(2)} T_n(x)$ with

$$c_n b_n^{(2)} = \sum_{p=n+2}^{N} p(p^2 - n^2) a_p, \quad (p + n \text{ even}).$$

The Chebyshev expansion for the square wave $[-1, 1]$ is

$$\theta_{Ch}(x) = \frac{4}{\pi} \left[ T_1(x) - \frac{T_3(x)}{3} + \frac{T_5(x)}{5} - \frac{T_7(x)}{7} + \cdots \right].$$

Note that

$$\theta_{Ch}(1) = \frac{4}{\pi} \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right] = 1.$$
Figure 3: The Chebyshev expansion (red) avoids the Gibbs phenomenon for discontinuous jumps at boundaries (though not on the interior).

**Gibbs Phenomenon for the Square Wave**

The Fourier approximation for the square wave on $[-\pi, \pi]$ with $N+1$ terms is

$$\theta_N(x) = \sum_{n=0}^{N} \frac{4}{(2n+1)\pi} \sin((2n+1)x).$$

The first maximum of $\theta_N$ to the right of the origin is at $\bar{x}$ where

$$\theta'_N(\bar{x}) = \sum_{n=0}^{N} \frac{4}{\pi} \cos((2n+1)\bar{x}) = \frac{2}{\pi} \frac{\sin(2(N+1)\bar{x})}{\sin(\bar{x})} = 0$$

which implies $\bar{x} = \pi/(2(N+1))$. The Gibbs overshoot then is

$$\lim_{N \to \infty} \theta_N(\bar{x}) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{4}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{2(N+1)}\right) =$$

$$\frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(y)}{y} dy \approx \frac{2}{\pi} \cdot 1.85194 \approx 1.17898$$

where we set

$$y = \frac{(2n+1)\pi}{2(N+1)}, \quad dy = \frac{2\pi \Delta n}{2(N+1)}, \quad \frac{dy}{y} = \frac{2\Delta n}{2n+1}.$$