A Comparison of Modern Finite-Volume Hyperbolic Methods for Nonlinear Conservation Laws, with Applications to Astrophysical Jets and Semiconductor Devices

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Euler Equations of Gas Dynamics with Radiative Cooling

\[ \frac{\partial}{\partial t} (mn) + \frac{\partial}{\partial x_i} (mnu_i) = 0 \]

\[ \frac{\partial}{\partial t} (mnu_j) + \frac{\partial}{\partial x_i} (mnu_i u_j - P_{ij}) = 0 \]

\[ \frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i} (u_i W - u_j P_{ij}) = -n^2 \Lambda(T) \]

Stress tensor & energy density

\[ P_{ij} = -nT \delta_{ij} \]

\[ W = \frac{3}{2} nT + \frac{1}{2} mnu^2 \]
Diagram of HH 30 Circumstellar Disk & Jet

Accretion Disk

Proto-Star

Jet
Jets from Young Stars

PRC95-24a · ST ScI OPO · June 6, 1995
C. Burrows (ST ScI), J. Hester (Az State U.), J. Morse (ST ScI), NASA
Jets from Young Stars • HH1/HH2
HST • WFPC2
PRC95-24c • ST Sci OPO • June 6, 1995 • J. Hester (AZ State U.), NASA
The NT Scheme

1D scalar conservation law $u_t + f(u)_x = 0$

Nessyahu-Tadmor (NT) scheme derived from the Lax-Friedrichs scheme

$$u_{j + \frac{1}{2}}(t+\Delta t) = \frac{1}{2}(u_j(t)+u_{j+1}(t)) - \lambda [f(u_{j+1}(t)) - f(u_j(t))], \quad \lambda = \frac{\Delta t}{\Delta x}$$

Replace piecewise constant solutions with piecewise linear approximations

$$L_j(x, t) = u_j(t) + (x - x_j) \frac{u_j'}{\Delta x}, \quad x_j - \frac{1}{2} \leq x \leq x_j + \frac{1}{2}$$
NT central scheme:

\[
u_j \left( t + \frac{\Delta t}{2} \right) = u_j - \frac{1}{2} \lambda f'_j
\]

\[
u_{j+\frac{1}{2}}(t + \Delta t) = \frac{1}{2}(u_j(t) + u_{j+1}(t)) + \frac{1}{8}(u'_j - u'_{j+1}) - \lambda \left[ f \left( u_{j+1} \left( t + \frac{\Delta t}{2} \right) \right) - f \left( u_j \left( t + \frac{\Delta t}{2} \right) \right) \right]
\]

CFL condition

\[
\lambda \max_x \left| \frac{\partial f}{\partial u} \right| < \frac{1}{2}
\]

NT scheme is 2nd-order accurate & TVD if (e.g.)

\[
u'_j = \text{minmod} \left\{ \theta \Delta u_{j-\frac{1}{2}}, \frac{1}{2}(u_{j+1} - u_{j-1}), \theta \Delta u_{j+\frac{1}{2}} \right\}
\]

\[
f'_j = \text{minmod} \left\{ \theta \Delta f_{j-\frac{1}{2}}, \frac{1}{2}(f_{j+1} - f_{j-1}), \theta \Delta f_{j+\frac{1}{2}} \right\}
\]

where \(\theta \in [1, 2]\) & \(\Delta u_{j+\frac{1}{2}} = u_{j+1} - u_j\)
Jet Parameters

<table>
<thead>
<tr>
<th>jet</th>
<th>ambient</th>
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</thead>
<tbody>
<tr>
<td>$\gamma = 5/3$</td>
<td>$\gamma = 5/3$</td>
</tr>
<tr>
<td>$\rho_j = 500 \text{ H/cm}^3$</td>
<td>$\rho_a = 50 \text{ H/cm}^3$</td>
</tr>
<tr>
<td>$u_j = 300 \text{ km/s}$</td>
<td>$u_a = 0$</td>
</tr>
<tr>
<td>$T_j = 1000 \text{ K}$</td>
<td>$T_a = 10,000 \text{ K}$</td>
</tr>
<tr>
<td>$c_j = 4 \text{ km/s}$</td>
<td>$c_a = 13 \text{ km/s}$</td>
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</tbody>
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Computational Units

density $\bar{\rho} = 100 \text{ H/cm}^3$
velocity $\bar{v} = 10 \text{ km/s}$
length $\bar{l} = 10^{11} \text{ km}$
⇒ temperature $\bar{T} = 1.2 \times 10^4 \text{ K}$
Density at time $t = 0.4$

Pressure at time $t = 0.4$

Temperature at time $t = 0.4$

NTK METHOD ($u = 2.0$, $\theta = 1.1$)
Density at time $t = 0.4$

Pressure at time $t = 0.4$

Temperature at time $t = 0.4$

GODUNOV METHOD ($u = 2.0$, CFL = 0.1, w/o cooling)
Density at time $t = 0.4$

Pressure at time $t = 0.4$

Temperature at time $t = 0.4$

WENO METHOD ($u = 2.0$, CFL = 0.6)
Density at time $t = 0.07$

Pressure at time $t = 0.07$

Temperature at time $t = 0.07$

NTK METHOD ($u = 30$, $\theta = 1.1$, CFL = .25)
Density at time $t = 0.07$

Pressure at time $t = 0.07$

Temperature at time $t = 0.07$

$\alpha = 2.5$, CFL = .45, WENO–LF (with cooling by exact)
WENO METHOD ( \( u = 30.0 \), CFL = 0.5) with BLOBS
Classical Hydrodynamic Model =

**Electro-Gas Dynamics**

\[
\frac{\partial}{\partial t} (mn) + \frac{\partial}{\partial x_i} (mn u_i) = 0
\]

\[
\frac{\partial}{\partial t} (mn u_j) + \frac{\partial}{\partial x_i} (mn u_i u_j) + \frac{\partial}{\partial x_j} (nT) = -n \frac{\partial V}{\partial x_j} - \frac{mn u_j}{\tau_p}
\]

\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial x_i} (u_i (W + nT) + q_i) = -nu_i \frac{\partial V}{\partial x_i} \left( \frac{W - \frac{3}{2} nT_0}{\tau_w} \right)
\]

\[
\nabla \cdot (\varepsilon \nabla V) = e^2 (N - n), \quad V = -e\phi, \quad -eE = e\nabla \phi = -\nabla V
\]

**Energy density**

\[W = \frac{3}{2} nT + \frac{1}{2} mn u^2, \quad \left( \gamma = \frac{5}{3} \right)\]

**Heat flux**

\[q = -\kappa n \nabla T, \quad \kappa = \kappa_0 \mu_0 T_0\]
$n^+ - n - n^+ \text{ Diode}$

Figure 1: Hydrodynamic vs. DAMOCLES Monte Carlo simulation of an electron shock wave in a 1 micron channel at 300 K.
Figure 2: CLAWPACK simulation of an electron shock wave in a 0.25 micron channel at 77 K.
Figure 3: Electron velocity in $10^7$ cm/s using CLAWPACK (solid) vs. the NTK central scheme (dotted—the dots are solution values).
Figure 4: Closeup of the electron shock wave in velocity.
2D MESFET (0.4 micron channel) simulation of electron density using NTK (with Anne Gelb)