Numerical Methods for Boundary Value Problems

BVPs are usually formulated for $y(x)$. Along the $x$ axis, allocate gridpoints $x_i, i = 0, \ldots, N$. Boundary conditions will be imposed at $x_0$ and $x_N$.

First and Second Derivative Matrices

First and second derivatives at the interior gridpoints $x_1, \ldots, x_{N-1}$ will be computed from solution values $y = [y_1, \ldots, y_{N-1}]$ and boundary conditions $y_0$ and $y_N$ by

$$y' = D^{(1)} y$$

and

$$y'' = D^{(2)} y.$$ 

For $N = 10$ (boundary conditions are discussed below),

$$D^{(1)} = \frac{1}{2h} \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1
\end{bmatrix}$$

$$D^{(2)} = \frac{1}{h^2} \begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{bmatrix}.$$
Linear BVPs

As an example, let’s discretize the linear BVP
\[ y'' + 2y' + y = 0, \quad y(0) = 1, \quad y(1) = 0. \]

We can compare our computed solution against the exact solution \( y(x) = (1 - x)e^{-x} \). The discrete equations for \( y = [y_1, \ldots, y_{N-1}] \) are
\[ D^{(2)}y + 2D^{(1)}y + y \equiv Ay = b \]
where
\[ b = [-y_0/h^2 + y_0/h, 0, \ldots, 0, -y_N/h^2 - y_N/h] \]
incorporates the coupling to the boundary conditions. The solution to the BVP is given by numerically solving \( Ay = b \).

Nonlinear BVPs and Newton’s Method

As a nonlinear example, let’s compute the solution to the boundary layer equation \((0 < \epsilon \ll 1)\)
\[ \epsilon y'' + 2y' + e^y = 0, \quad y(0) = 0, \quad y(1) = 0. \]

The discrete equations for \( y = [y_1, \ldots, y_{N-1}] \) are
\[ \epsilon D^{(2)}y + 2D^{(1)}y + e^y \equiv F(y) = 0. \]

To solve this system of nonlinear equations, we make a guess for \( y \) and then iterate using Newton’s method until the norm of the residual \( ||F|| \leq \epsilon_R \):
\[ y^{(k+1)} = y^{(k)} + \Delta y, \quad y^{(0)} = 0, \quad k = 0, 1, 2, \ldots \]
\[ 0 = F\left(y^{(k+1)}\right) \approx F\left(y^{(k)}\right) + J\Delta y \]
where the Jacobian
\[ J = \frac{\partial F}{\partial y}\bigg|_{y^{(k)}} = \epsilon D^{(2)} + 2D^{(1)} + \text{diag}\{e^y\}. \]

Solving for the corrections \( \Delta y \), we get
\[ J\Delta y = -F, \quad y \leftarrow y + \Delta y. \]
A uniform global approximation to $y(x)$ solving Eq. (⋆) can be derived for $\epsilon \to 0$:

$$y(x) = \ln \left( \frac{2}{x + 1} \right) - \ln(2) \exp \left( -\frac{2x}{\epsilon} \right).$$

In this limit, the width of the boundary layer (location of the maximum of $y$) is given by

$$B = -\frac{1}{2} \epsilon \ln(\epsilon).$$

**Derivation of the uniform global approximation**

(i) In the outer laminar region, $y$ is changing slowly so we can neglect $\epsilon y''$ in comparison with the other two terms in (⋆):

$$2y'_\text{out} + \exp(y_{\text{out}}) = 0, \quad y_{\text{out}}(x) = \ln \left( \frac{2}{x + \epsilon} \right) = \ln \left( \frac{2}{x + 1} \right)$$

to satisfy the boundary condition $y(1) = 0$.

(ii) In the inner boundary layer, $y$ is changing rapidly. Set

$$y(x) = y_{\text{in}}(X) = y_{\text{in}} \left( \frac{x}{\epsilon} \right).$$

Then to leading order $\epsilon^{-1}$ in $\epsilon$ (noting that $d/dx = \epsilon^{-1} d/dX$),

$$\frac{d^2 y_{\text{in}}}{dX^2} + 2 \frac{d y_{\text{in}}}{dX} = 0$$

$$y_{\text{in}}(X) = a + be^{-2X} = a(1 - e^{-2X})$$

to satisfy the boundary condition $y(0) = 0$. By requiring that $y_{\text{in}}(X) = \ln(2)$ as $X \to \infty$ to match $y_{\text{out}}$ as $x \to 0$, we obtain

$$y_{\text{in}}(X) = \ln(2)(1 - e^{-2X}).$$

(iii) **Asymptotic matching.** The uniform global approximation is

$$y(x) = \ln \left( \frac{2}{x + 1} \right) - \ln(2) \exp \left( -\frac{2x}{\epsilon} \right)$$

since it agrees with $y_{\text{in}}$ inside the boundary layer and with $y_{\text{out}}$ in the laminar region.