Problem Set 3

Due: Mon Feb 25
Reading: LeVeque: Chapters 1–3 + 5–6 + 9–11

(1) Derive the entropy advection equation \( s_t + us_x = 0 \) from the Euler equations and the expression for the entropy \( s = c_V \ln(p/\rho^\gamma) \) of a polytropic gas. This equation is valid only away from discontinuities. \textit{Hint:} Start with the 1D Euler equations with third equation \( s_t + us_x = 0 \) and then derive the energy conservation equation \( E_t + (u(E + P))_x = 0 \).

[Note (but don’t show): The entropy density \( S = \rho s/m \) satisfies the conservation law \( S_t + (uS)_x = 0 \).]

(2) Derive the Hugoniot relation for a shock (0 is ahead of and 1 is behind the shock)

\[
e_1 - e_0 = \frac{P_0 + P_1}{2} \left( \frac{\rho_0}{\rho_1} - \frac{1}{\rho_1} \right)
\]

where the energy density \( E = \rho e + \frac{1}{2} \rho u^2 \) and \( e \) is the specific internal energy. Then derive

\[
\left( \frac{1}{\rho_1} - \frac{\mu^2}{\rho_0} \right) P_1 = \left( \frac{1}{\rho_0} - \frac{\mu^2}{\rho_1} \right) P_0
\]

where \( \mu^2 = (\gamma - 1)/(\gamma + 1) \).

(3) Show that the solution to the Riemann problem

\[
\rho_L = 2, \ u_L = 1, \ p_L = 3
\]

\[
\rho_R = 1, \ u_R = 0, \ p_R = 1
\]

with \( \gamma = 1.5 \) is a single shock wave. Find the shock speed \( s_0 \). With a wall BC at \( x = 1 \), analytically calculate the solution after reflection of the shock wave. \textit{Hint:} Show that the exact reflected shock solution is

\[
\rho = 3.6, \ u = 0, \ p = 7.5, \ s = -1.25
\]

(4) [modified LeVeque 6.1] Compute the eigenvectors of

\[
A = \begin{bmatrix}
0 & -1 \\
-c^2 & 0
\end{bmatrix}
\]

and use them to decouple \( q_t + Aq_x = 0 \) into a pair of scalar equations. Solve these scalar equations and then transform back to compute the D’Alembert
solution \( u(x, t) \) of the original wave equation \( u_{tt} - c^2 u_{xx} = 0 \).

(5) [LeVeque 6.2] Rewrite the 2D wave equation

\[
u_{tt} - c^2 (u_{xx} + u_{yy}) = 0
\]

as a first-order system \( q_t + Aq_x + Bq_y = 0 \) where \( q = [v, w, \varphi] \) and \( v = u_x, w = u_y \), and \( \varphi = u_t \).

(6) [LeVeque 6.4] Verify that

\[
\rho(x, t) = \bar{\rho}(x - u_0 t), \quad u(x, t) = u_0, \quad P(x, t) = P_0, \quad e(x, t) = e_0
\]

satisfy the Euler equations, where \( \bar{\rho}(x) \) is any density distribution. Recall \( E = \rho e + \frac{1}{2} \rho u^2 \). Is this solution isentropic?

[Note: Newton assumed that sound wave propagation in air was isothermal and predicted that the soundspeed \( c = \sqrt{P/\rho} \), but it is actually a reversible adiabatic process = isentropic \( \rightarrow P = A \rho^\gamma \) with \( c = \sqrt{\gamma P/\rho} \). Adiabatic means no gain or loss of heat, so reversible adiabatic implies that the change in heat \( dQ = T ds = 0 \), where \( s \) is the entropy. (An irreversible process would have \( dQ < T ds \)).]