Problem Set 6

Diss: Mon Nov 5
Reading: LeVeque: Chapters 10, E.1, E.2, E.3

Please label with APM 522, PS6, and your name; write-up the problems neatly and succinctly (≤ 2 pages/problem, best if ≤ 1 page/problem) and in numerical order; and write (2) — if problem (2) is not attempted. No need to restate problem. Derive your answers!

1) In the modified PDE for the Lax-Wendroff method for \( u_t + cu_x = 0 \), derive the coefficient \( \beta = \frac{\Delta t^2}{6} (r^2 - 1) \) of numerical dispersion in \( u_t + cu_x = \beta u_{xxx} \).

2) Show that the two-step Lax-Wendroff method reduces to the original Lax-Wendroff scheme for \( u_t + Au_x = 0 \).

3) Show that Burgers’ equation \( u_t + uu_x = \nu u_{xx} \) with \( u(x, t = 0) = u_l, x < 0 \) and \( u(x, t = 0) = u_r, x > 0 \) has a traveling wave solution of the form \( u(x, t) = w(x - st) \) by deriving an ODE for \( w \) and showing that the ODE is solved by

\[
   w(y) = u_r + \frac{1}{2} (u_l - u_r) \left[ 1 - \tanh \left( \frac{u_l - u_r}{4\nu} y \right) \right]
\]

where \( s = (u_l + u_r) / 2 \).

4) Prove that the Lax-Friedrichs (LF) method is positivity-preserving (i.e., if \( u^n_j > 0 \) for all \( j \), then \( u^{n+1}_j > 0 \) for all \( j \)) for Burgers’ equation \( u_t + (u^2/2)_x = 0 \).

The LF discretization of Burgers’ equation is

\[
   u^{n+1}_j = \frac{1}{2} \left( u^n_{j-1} + u^n_{j+1} \right) - \frac{\Delta t}{4\Delta x} \left( (u^n_{j+1})^2 - (u^n_{j-1})^2 \right)
\]

Hint: For stability, \( \Delta t \leq \Delta x / \max_{i}\{|u^n_i|\} \). Note: Lax-Friedrichs is also positivity-preserving for gas dynamics, unlike upwind schemes.

5) Show that the upwind discretization of Burgers’ equation \( u_t + uu_x = 0 \) for \( u(x, t) > 0 \)

\[
   u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x} u^n_j (u^n_j - u^n_{j-1})
\]
is nonconservative, while the upwind discretization of Burgers’ equation \( u_t + (\frac{1}{2} u^2)_x = 0 \) for \( u(x, t) > 0 \)

\[
u_{j}^{n+1} = u_{j}^{n} - \frac{\Delta t}{2\Delta x} (u_{j}^{n})^{2} - (u_{j-1}^{n})^{2}\]

is conservative, by showing whether or not the scheme can be put into the form

\[
u_{j}^{n+1} = u_{j}^{n} - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}).\]