APM 522 Numerical Methods for Partial Differential Equations  
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Problem Set 5

Reading: LeVeque: Chapters 10, E.1, E.2, E.3

Please label with APM 522, PS5, and your name; write up the problems neatly and succinctly (≤ 2 pages/problem, best if ≤ 1 page/problem) and in numerical order; and write (2) — if problem (2) is not attempted. No need to restate problem. Derive your answers!

(1) Derive the modified PDE for the Lax-Friedrichs method for $u_t + cu_x = 0$. Find the coefficient $D_n \sim \{\Delta t, \Delta x\}$ of numerical diffusion in $u_t + cu_x = D_n u_{xx}$. Note that $D_n \geq 0$ iff the Courant number $r = c\Delta t/\Delta x \leq 1$.

(2) (a) Show that the Lax-Friedrichs method is first order (using the definition of the LTE) and (b) conditionally stable (using von Neumann stability analysis) for $u_t + cu_x = 0$. Hint for stability analysis: Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$.

(3) Show that Lax-Friedrichs is conservative by verifying that the numerical flux function

$$F_{i+\frac{1}{2}} = \frac{1}{2} (f(w_i) + f(w_{i+1})) - \frac{\Delta x}{2\Delta t} (w_{i+1} - w_i)$$

correctly produces the Lax-Friedrichs method for $w_t + f(w)_x = 0$.

(4) Using von Neumann stability analysis, show downwind is unconditionally unstable for $u_t + cu_x = 0$. Hint: Show $|G(k)|^2 = G^*(k)G(k) > 1$ for any value of $r > 0$.

(5) Using von Neumann stability analysis, show that Lax-Wendroff is stable for $u_t + cu_x = 0$ as long as the CFL condition $r \leq 1$ is satisfied. Hint: Show $|G(k)|^2 = G^*(k)G(k) \leq 1$ iff $r \leq 1$. 