**Problem Set 4**

Due: Fri Oct 20  
*Reading:* LeVeque: Chapters 3, 4.1–4.3, E.1, E.2

*Please label with APM 522, PS4, and your name; write-up the problems neatly and in numerical order; and write (2) omitted if problem (2) is not attempted. No need to restate problem. Derive your answers! If you typeset the whole assignment or write it up on a tablet, you may email me a PDF instead of turning in a hard copy. Please do not send me photographs of handwritten assignments.*

1. Show that the Jacobi spectral radius $\mu = \cos(\pi h)$ for Laplace’s equation on the unit square with second-order accurate central differences. Write-up only the 1D version. *Hint:* In 1D, set $A = \text{tridiag}[-1, 2, -1]$. Then the iteration matrix $B = \frac{1}{2} \text{tridiag}[1, 0, 1]$. Then show that $Bv = \cos(\pi h)v$ where the 1D eigenvector
   
   $$v = [\sin(\pi h), \sin(2\pi h), \cdots, \sin(n\pi h)].$$

   Note that here $h = 1/(n + 1)$.

2. (a) For SOR/SUR, show that $\det\{B\} = (1 - \omega)^n$, $0 < \omega < 2$. *Hint:* $B$ is the product of triangular matrices.

   (b) Derive the equation for the SOR $\omega_{opt}$, assuming Young’s formula applied to the spectral radii:
   
   $$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2.$$

   Here $\mu$ is the spectral radius for the Jacobi iteration method and $\lambda$ is the spectral radius for the SOR iteration method. *Hint:* Set $\lambda = \omega - 1$ and minimize $\lambda$ (using the quadratic formula).

   (c) Show that for Laplace’s equation on the unit square, the SOR $\lambda = (1 - \sin \pi h)/(1 + \sin \pi h)$.

3. $2 \times 2$ SOR example. Calculate the first two iterates $x_1$ and $x_2$ for SOR
with \( x_0 = (0, 0) \) for \( Ax = b \) with

\[
A = \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}, \quad b = \begin{bmatrix}
3 \\
-3
\end{bmatrix}
\]

\[
M_{\text{SOR}} = \begin{bmatrix}
\frac{2}{\omega} & 0 \\
-1 & \frac{2}{\omega}
\end{bmatrix}, \quad B_{\text{SOR}} = \begin{bmatrix}
1 - \omega & \frac{\omega}{\omega^2} \\
\frac{\omega}{\omega^2}(1 - \omega) & (1 - \frac{\omega}{\omega^2})^2
\end{bmatrix}
\]

\[
\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.0718, \quad \lambda_1 = \lambda_2 = \omega_{\text{opt}} - 1 = \rho_{\text{SOR}} \approx 0.0718.
\]

The exact solution is \( x = (1, -1), x^J_2 = (3/4, -3/4) \), and \( x^{GS}_2 = (9/8, -15/16) \). Calculate \( \|e^J_2\|_1, \|e^{GS}_2\|_1, \) and \( \|e^{SOR}_2\|_1 \). Note that the SOR \( x_2 \) is much closer to the exact solution.

(4) Use conjugate gradient on the steepest descent problem we did in class:

\[
A = \begin{bmatrix}
4 & -2 \\
-2 & 2
\end{bmatrix}, \quad b = \begin{bmatrix}
-2 \\
2
\end{bmatrix}, \quad x_0 = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad x = \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

(5) Compute the first two conjugate gradient iterates \( x_1 \) and \( x_2 \) with \( x_0 = (0, 0) \) with and without preconditioning to the solution \( x = (0, 1) \) of \( Ax = b \):

\[
A = \begin{bmatrix}
9 & 1 \\
1 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \quad M = \begin{bmatrix}
9 & 0 \\
0 & 1
\end{bmatrix}.
\]

Calculate \( \|e_1^{CG}\|_1 \) and \( \|e_1^{PCG}\|_1 \). Note that while both CG and PCG give the exact \( x \) in two steps \( x_2 = (0, 1) \), PCG gives a much better \( x_1 \).

(6) Compare the number of iterations required for convergence for SOR (laplace3.m), CG (laplace4.m), and PCG (laplace5.m) for the model Laplace problem on three grids: \( 100 \times 100, 200 \times 200, \) and \( 400 \times 400 \). Turn in just a Table. Note that for convergence of the iterative methods, \( \epsilon = 10^{-5}h^2 \).

<table>
<thead>
<tr>
<th>solver</th>
<th>100 × 100</th>
<th>200 × 200</th>
<th>400 × 400</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOR</td>
<td>411</td>
<td>876</td>
<td>1858</td>
</tr>
<tr>
<td>CG</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCG</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Number of iterative sweeps for the model Laplace problem solved in MATLAB on three grids.