Problem Set 2

Reading: LeVeque: Chapters 1, 9, E.1, E.2

Please label with APM 522, PS2, and your name; write up the problems neatly and succinctly (≤ 2 pages/problem, best if ≤ 1 page/problem) and in numerical order; and write (2) — if problem (2) is not attempted. No need to restate problem. Derive your answers!

(1) Prove that the forward Euler method for the heat equation $u_t = u_{xx}$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2} \left( u_{i+1}^n - 2u_i^n + u_{i-1}^n \right)$$

is (a) first-order accurate and (b) conditionally stable. What is the restriction on $\Delta t$?

(2) Prove that the TR method is A-stable but not L-stable for the heat equation $u_t = u_{xx}$. (Note that the trapezoidal rule (TR) method for the heat equation is also known as Crank-Nicolson.)

(3) Show that the second-order accurate central discretization

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{\Delta x^2} \left( \frac{1}{4} (D_{i+1} - D_{i-1})(u_{i+1} - u_{i-1}) + D_i(u_{i+1} - 2u_i + u_{i-1}) \right)$$

of $u_t = D_x u_x + Du_{xx}$ is nonconservative. (You can use $Q^n = \sum_{i=0}^{N} u_i^n \Delta x$ and $N = 4$. Then show that the terms multiplying $u_2$ in $(Q^{n+1} - Q^n)/\Delta t$ do not cancel unless $D$ is constant.)

(4) (a) Using the trapezoidal rule integration method to approximate

$$Q(t) = \int u(x, t)dx \approx Q^n = \sum_{i=0}^{N-1} \frac{u_i^n + u_{i+1}^n}{2} \Delta x$$

show that discretizations of $u_t + f_x = 0$ of the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right)$$
are conservative. (b) For nonlinear diffusion, \( f(u) = -D(u)u_x \) and \( F_{i+\frac{1}{2}} = D_{i+\frac{1}{2}}(u_{i+1} - u_i)/\Delta x \). Show that if homogeneous Neumann boundary conditions \( u_x(a,t) = 0 = u_x(b,t) \) are imposed, \( Q \) is constant in time.

(5) Derive the Fourier solution to the initial/boundary value problem for the heat equation with homogeneous Neumann boundary conditions

\[
 u_t = u_{xx}, \quad u_x(0,t) = 0 = u_x(\pi,t), \quad u(x,t=0) = u_0(x)
\]

by making a Fourier cosine expansion (since it automatically satisfies the boundary conditions)

\[
 u(x,t) = \sum_{n=0}^{\infty} a_n(t) \cos(nx)
\]

and solving for \( a_n(t) \).