Problem Set 2

Reading: LeVeque: Chapters 1, 9, E.1, E.2

Please label with APM 522, PS2, and your name; write up the problems neatly and succinctly (≤ 2 pages/problem, best if ≤ 1 page/problem) and in numerical order; and write (2) — if problem (2) is not attempted. No need to restate problem. Derive your answers!

(1) Prove that the forward Euler method for the heat equation \( u_t = u_{xx} \)

\[
u_{i}^{n+1} = u_{i}^{n} + \frac{\Delta t}{\Delta x^2} \left(u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}\right)
\]

is (a) first-order accurate and (b) conditionally stable. What is the restriction on \( \Delta t \)?

(2) Prove that the TR method is A-stable but not L-stable for the heat equation \( u_t = u_{xx} \).

(3) Show that the second-order accurate central discretization

\[
u_{i}^{n+1} = u_{i}^{n} + \frac{\Delta t}{\Delta x^2} \left(\frac{1}{4}(D_{i+1} - D_{i-1})(u_{i+1} - u_{i-1}) + D_i(u_{i+1} - 2u_{i} + u_{i-1})\right)
\]

of \( u_t = D_xu_x + Du_{xx} \) is nonconservative. (You can use \( Q^n = \sum_{i=0}^{N} u^n_i \Delta x \) and \( N = 4 \). Then show that the terms multiplying \( u_2 \) in \( (Q^{n+1} - Q^n)/\Delta t \) do not cancel unless \( D \) is constant.)

(4) (a) Using the trapezoidal rule integration method to approximate \( Q^n \approx Q(t) = \int u(x,t)dx \), show that discretizations of \( u_t + f_x = 0 \) of the form

\[
u_{i}^{n+1} = u_{i}^{n} - \frac{\Delta t}{\Delta x} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)
\]

are conservative. (b) For nonlinear diffusion, \( f(u) = -D(u)u_x \) and \( F_{i+\frac{1}{2}} = D_{i+\frac{1}{2}}(u_{i+1} - u_{i})/\Delta x \). Show that if homogeneous Neumann boundary conditions \( u_x(a,t) = 0 = u_x(b,t) \) are imposed, \( Q \) is constant in time.
(5) Derive the Fourier solution to the initial/boundary value problem for the heat equation with homogeneous Neumann boundary conditions

\[ u_t = u_{xx}, \quad u_x(0, t) = 0 = u_x(\pi, t), \quad u(x, t = 0) = u_0(x) \]

by making a Fourier cosine expansion (since it automatically satisfies the boundary conditions)

\[ u(x, t) = \sum_{n=0}^{\infty} a_n(t) \cos(nx) \]

and solving for \( a_n(t) \).