Problem Set 1

Due: Wed Aug 29
Reading: LeVeque: Chapters 1, 9, E.1, E.2; MATLAB: Chapter 1.1, 1.7 (IEEE floating point)

Please label with APM 522, PS1, and your name; write up the problems neatly and succinctly (≤ 2 pages/problem, best if ≤ 1 page/problem) and in numerical order; and write (2) — if problem (2) is not attempted. No need to restate problem or to show algebra — but do show steps. Derive your answers! If you typeset the whole assignment, you may email me a PDF instead of turning in a hardcopy.

(1) Suppose a double precision floating point number is stored on a computer using 64 bits in the following way: sign 1 bit, exponent 8 bits, and mantissa 55 bits. A given real number $r$ is written as

$$r = \pm m2^n$$

where the mantissa $m$ satisfies $1/2 \leq m < 1$ and $-128 \leq n \leq 127$. Give the following numbers in both base 2 and base 10 scientific notation:

(a) What is the largest positive number $\text{realmax}$ that can be stored?

(b) What is the smallest positive number $\text{realmin}$ that can be stored?

(c) What is the machine epsilon $\epsilon_M$ (take the leading 1 in $m$ to be a phantom)?

For problems (2) and (3), write formulas for the leading error term as

$$\frac{du}{dt} \approx \frac{u^{n+1} - u^n}{\Delta t} = u' + \frac{u''}{2} \Delta t + \cdots$$

or

$$\left( \frac{d^2 f}{dx^2} \right)_i \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} = f''_i + \frac{\Delta x^2}{12} f^{(4)}_i + \cdots$$
(2) (a) Verify that the three-point central difference formulas for $df/dx$ and $d^2f/dx^2$ are second-order accurate.
(b) Verify that the one-sided difference formula
\[ \frac{du}{dt} \approx \frac{u^{n+1} - u^n}{\Delta t} \]
is first-order accurate.

(3) Verify that the approximation
\[ \left( \frac{df}{dx} \right)_j \approx \frac{1}{\Delta x} \left[ -\frac{1}{12}f_{j+2} + \frac{2}{3}f_{j+1} - \frac{2}{3}f_{j-1} + \frac{1}{12}f_{j-2} \right] \]
is fourth-order accurate.

(4) Show that the general solution to the heat equation
\[ u(x, t) = \int_{-\infty}^{\infty} K(x - y, t)u_0(y)\,dy \]
where the fundamental solution or kernel
\[ K(x, t) = \frac{1}{\sqrt{4\pi \kappa t}} \exp \left\{ -\frac{x^2}{4\kappa t} \right\} \]
does in fact satisfy $u_t = \kappa u_{xx}$ with $u(x, t = 0) = u_0(x)$. \textit{Hint:} First show that $K$ satisfies the heat equation. Then argue that $u$ satisfies the heat equation. Finally show that $u$ satisfies the initial conditions $u(x, t = 0) = u_0(x)$.

(5) Show that the TR method for the heat equation $u_t = u_{xx}$
\[ u_{i+1}^n = u_i^n + \frac{\Delta t}{2\Delta x^2} \left( u_{i+1}^{n-1} - 2u_i^n + u_{i-1}^{n-1} + u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1} \right) \]
is second-order accurate, using the definition of the local truncation error: i.e., show that the global error $\tau = c_1 \Delta t^2 + c_2 \Delta x^2 + \cdots$, where “\ldots” means higher order terms in $\Delta t$ and $\Delta x$. (Fill in the values of $c_1$ and $c_2$.) \textit{Hint:} Use the facts that
\[ \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = u_{xx} + \frac{h^2}{12}u_{xxxx} + \cdots, \quad h \equiv \Delta x \]
and
\[ u_{xx}(t + \Delta t) = u_{xx} + \Delta t u_{xxt} + \frac{\Delta t^2}{2}u_{xxtt} + \cdots \]
Sample Solution

Verify the upwind difference formula (for flow to the right).

\[
\left( \frac{df}{dx} \right)_i \approx \frac{f_i - f_{i-1}}{h} = \\
\frac{1}{h} \left( f_i - \left( f_i - hf_i' + \frac{h^2}{2} f_i'' - \cdots \right) \right) = f_i' - \frac{h}{2} f_i'' + \cdots
\]