Mixed Partials

Ex. \( f(x,y) = \frac{x}{y} \)

\( f_x = \frac{1}{y}, \; f_y = -\frac{x}{y^2} \)

\( f_{xx} = 0, \; f_{xy} = -\frac{1}{y^2} = f_{yx}, \; f_{yy} = \frac{2x}{y^3} \)

If \( f(x,y) \) has continuous second derivatives, then \( f_{xy} = f_{yx} \)

If \( f_{xy} \) & \( f_{yx} \) are continuous on a disk containing \((a,b)\), then

\( f_{xy}(a,b) = f_{yx}(a,b) \)

Tangent Planes

\( z = f(x,y) \) Tangent plane at \((x_0, y_0, z_0)\) is

\[ z - z_0 = \left( \frac{\partial f}{\partial x} \right)_0 (x-x_0) + \left( \frac{\partial f}{\partial y} \right)_0 (y-y_0) \]

Normal vector to plane is \( \vec{N} = \left( \frac{\partial f}{\partial x}_0, \frac{\partial f}{\partial y}_0, -1 \right) \)

Ex. Find tangent plane to \( z = f(x,y) = 14 - x^2 - y^2 \) at \((x_0, y_0, z_0) = (1, 2, 9)\)

\( f_x = -2x, \; (f_x)_0 = -2 \)

\( f_y = -2y, \; (f_y)_0 = -4 \)

tangent plane \( z - 9 = -2(x-1) - 4(y-2) \) or

\[ z + 2x + 4y = 19 \]

Note: for the plane, \( \frac{\partial z}{\partial x} = -2 \) & \( \frac{\partial z}{\partial y} = -4 \) just like

surface at \((x_0, y_0, z_0)\)

paraboloid

\( (1, 2, 9) \)
Ex. Find tangent plane to sphere \( z^2 = 14 - x^2 - y^2 \) at \((x_0, y_0, z_0) = (1, 2, 3)\).

\[ z = \sqrt{14 - x^2 - y^2} \text{ upper hemisphere} \]

\[
\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{14 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{14 - x^2 - y^2}}
\]

\[
\left(\frac{\partial z}{\partial x}\right)_o = -\frac{1}{3}, \quad \left(\frac{\partial z}{\partial y}\right)_o = -\frac{2}{3}
\]

Tangent plane \( z - 3 = \frac{1}{3}(x - 1) - \frac{2}{3}(y - 2) \)

\[
\vec{N} = \left(\frac{1}{3}, \frac{-2}{3}, -1\right) = -\frac{1}{3}(1, 2, 3) \equiv -\frac{1}{3}\vec{N}_o
\]

from \( x_1 + 2(y - 2) + 3(z - 3) = 0 = \vec{N} \cdot (\vec{r} - \vec{r}_o) \)

Ex. Hyperboloid \( F(x, y, z) = x^2 + y^2 - z^2 = 1 \)

Note: Tangent plane to \( F(x, y, z) = c \) at \((x_0, y_0, z_0)\) is

\[
\left(\frac{\partial F}{\partial x}\right)_o (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_o (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_o (z - z_0) = 0
\]

Normal vector is \( \vec{N} = \left(\frac{\partial F}{\partial x}\right)_o, \left(\frac{\partial F}{\partial y}\right)_o, \left(\frac{\partial F}{\partial z}\right)_o \)

\( z = f(x, y) \) becomes \( F(x, y, z) = f(x, y) - z = 0 \) with \( \frac{\partial F}{\partial z} = -1 \)

For the hyperboloid, \( F_x = 2x, F_y = 2y, F_z = -2z \)

Tangent plane \( 2x_0 (x - x_0) + 2y_0 (y - y_0) - 2z_0 (z - z_0) = 0 \)

\( \vec{N} = (x_0, y_0, -z_0) \)
Differentials

\[ f(x,y) \quad df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = f_x \, dx + f_y \, dy \]

Ex

\[ V_{cyl} = \pi r^2 h \]

Is \( V_{cyl} \) more sensitive to a change in \( r \) or \( h \)?

\[ dV_{cyl} = 2\pi rh \, dr + \pi r^2 \, dh \]

Shell layer

Then

\[ dV_{cyl} = \frac{2\pi}{10} + \frac{\pi}{10} \]

Linear Approximation

\[ f(x,y) \approx f(x_0,y_0) + \left( \frac{\partial f}{\partial x} \right)_0 (x-x_0) + \left( \frac{\partial f}{\partial y} \right)_0 (y-y_0) \]

Ex

Find a linear approx to \( r = \sqrt{x^2+y^2} \)

\[ \Delta r \approx \frac{x}{r} \, \Delta x + \frac{y}{r} \, \Delta y \]

Ex

\[ F(x,y,z) = -x^2 - y^2 + z^2 = 0 \]

Find a linear approx to \( \Delta z \)

\[ \Delta F = 0 = -2x \, \Delta x - 2y \, \Delta y + 2z \, \Delta z \]

\[ \Delta z \approx \frac{x}{z} \, \Delta x + \frac{y}{z} \, \Delta y \] (same as previous Ex)